Thermodynamic Method for Generating Heterogeneous Initial Stress Conditions

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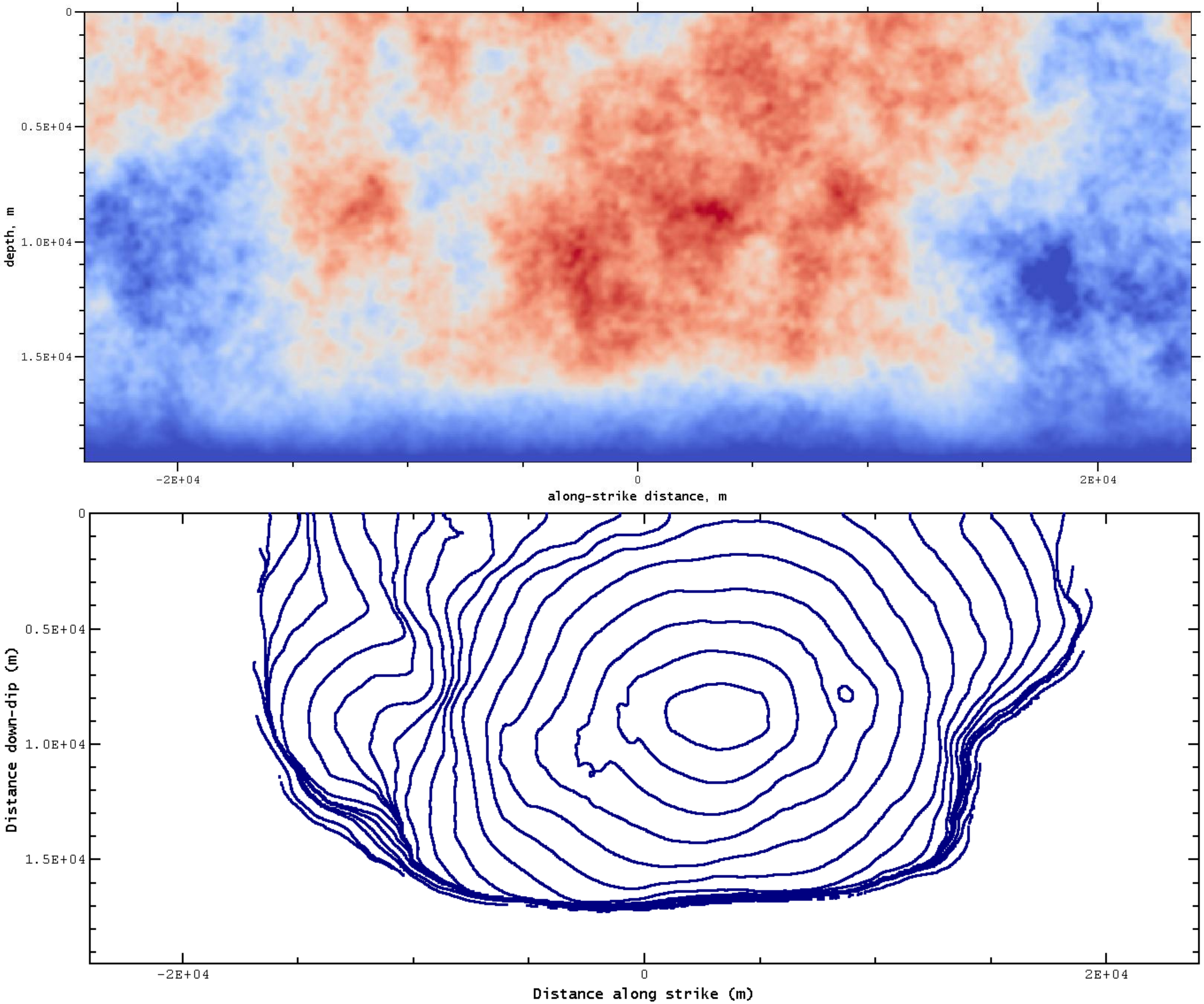

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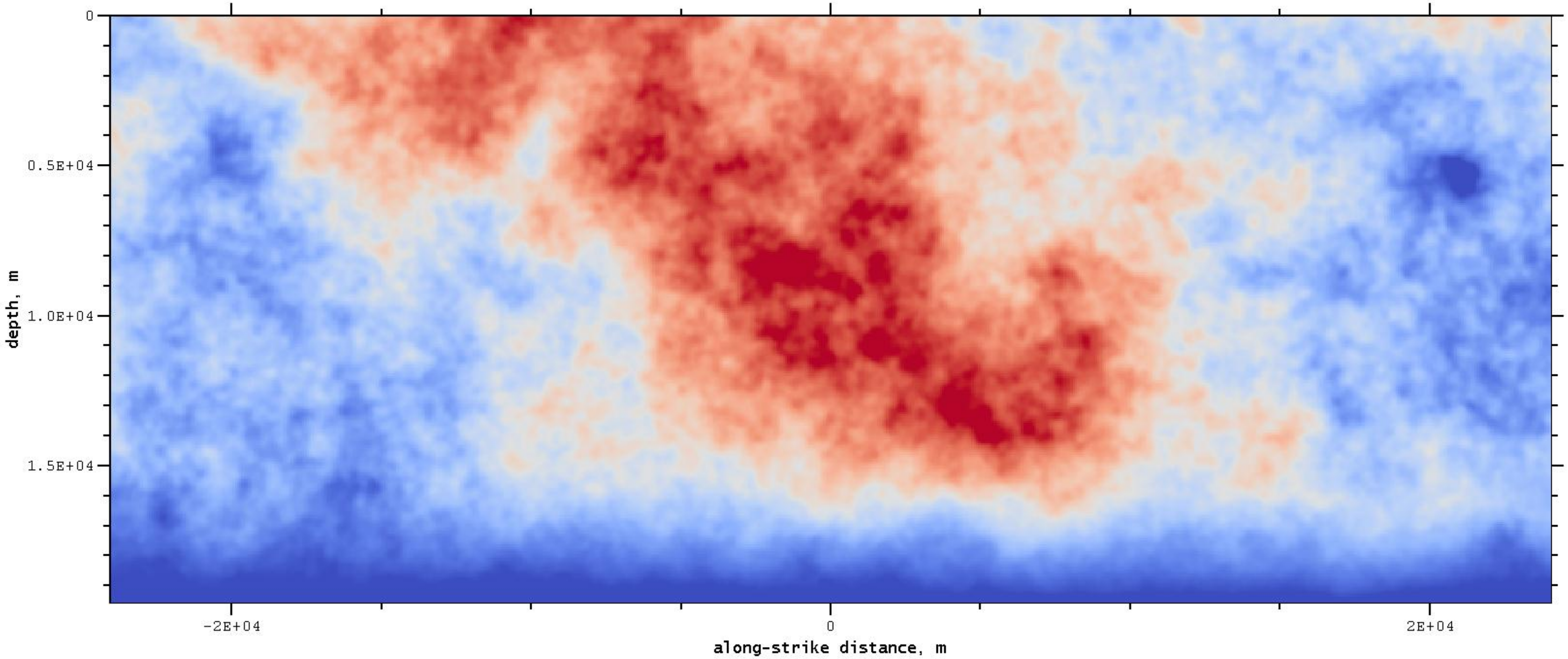
Distance along strike (m)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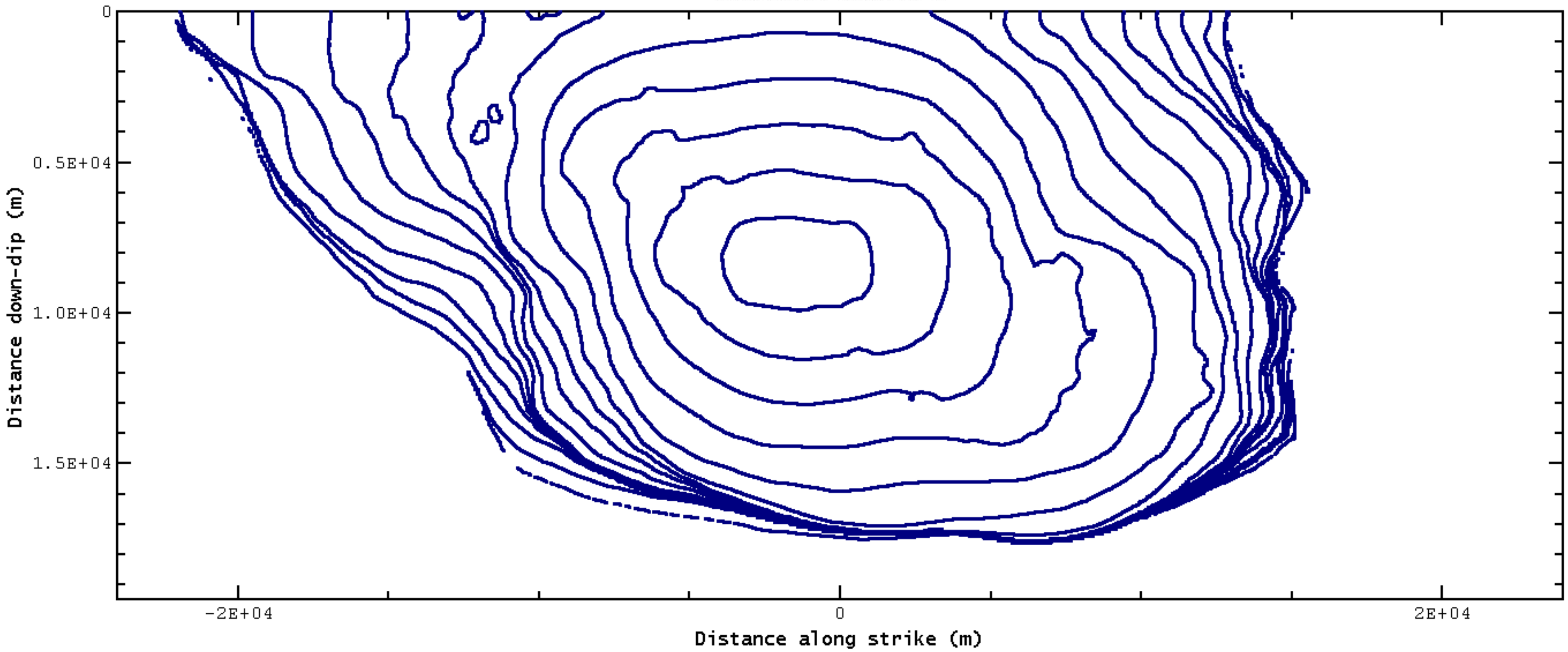
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Distance down-dip (m)
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Distance along strike (m)

Introduction

In 2010 we performed the 100 Runs project, in which we created a method to generate heterogeneous initial stress for dynamic rupture simulations. Like most existing methods, the 100 Runs method tries to emulate statistical properties of stress inferred from slip inversion models.

As a result, **we became dissatisfied with the currently-available methods for generating heterogeneous initial stress**.

- We do not have much faith in the slip inversion models. The slip inversion codes have limited resolution, particularly in the vertical direction, and their solutions are often non-unique.

- The methods try to draw conclusions, such as the functional form of the power spectrum at short wavelengths, which appear to exceed the resolution of the codes.

- Little attention is paid to the question of how to infer initial stress properties from final slip, and anything that doesn't affect final slip is ignored.

- The fact that a method mimics some statistical property of slip or stress does not necessarily imply it is a good method.

- The end result is a method whose parameters and design are, at best, weakly constrained. Construction of a method seems to involve many arbitrary choices.

In 2011, Ruth gave me the following assignment:

- Start with a clean sheet of paper, and devise a completely new method for generating heterogeneous initial stress conditions, which is not derived from slip inversion models.

- And do it on schedule.

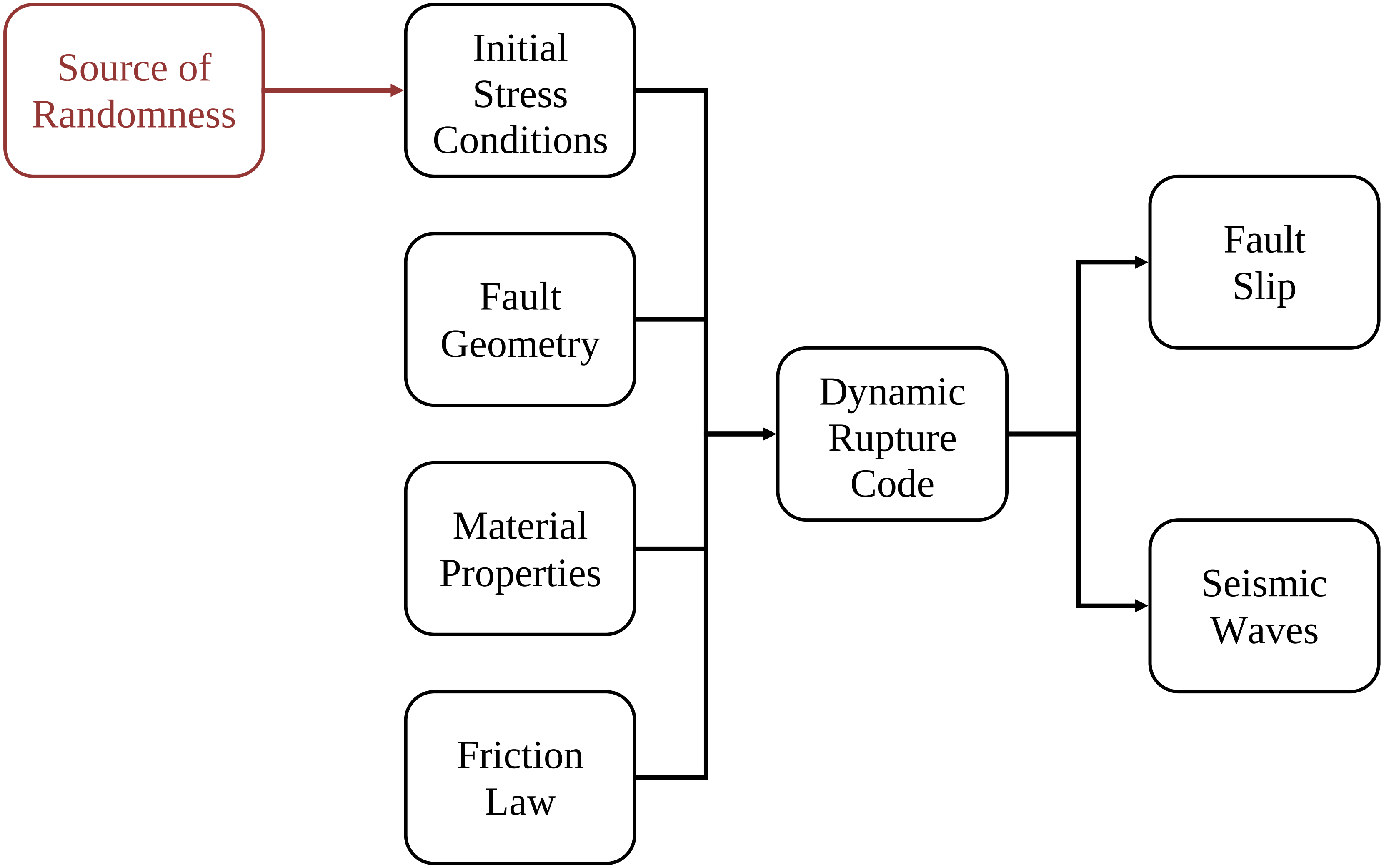
In carrying out this assignment, I have been guided by **Norm's advice:**

- "The goal is to do a better job than last time."

Our approach is to construct a new method for generating heterogeneous initial stress conditions, using concepts from thermodynamics and statistical mechanics, and in particular the 2D Ising model.

Previous work that uses statistical mechanics concepts to discuss stresses on faults:

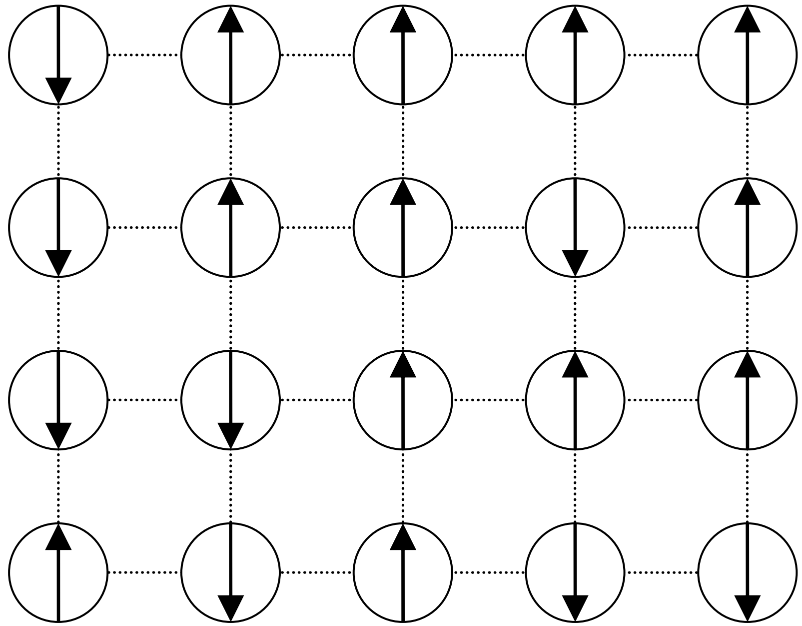
- R. Smalley Jr., D. Turcotte, and S. Solla, JGR 1985.
- Y. Huang, H. Saleur, C. Sammis and D. Sornette, Europhysics Letters 1998.
- J. Rundle, D. Turcotte, R. Shcherbakov, W. Klein, and C. Sammis, Reviews of Geophysics 2003.
- S. Abaimov, arxiv.org 2009.



We add randomness to dynamic rupture simulations, in order to make them more realistic. In last year's "100 Runs method," and most other methods, randomness is added to the initial stress conditions, yielding heterogeneous initial stress. The method presented here does the same thing.

A Ridiculously Short Review of Thermodynamics

2D Ising Model.



The *2D Ising model* is a model of a magnetic substance, such as iron. It demonstrates how a physical system can organize itself into large structures, such as magnetic domains. Each point on a 2D lattice holds a “spin” that can point either “up” or “down”. The spins act like small magnets, and each spin interacts only with its four nearest neighbors, so the total energy of the system is:

$$E = -\sum_{mn} K \, s_m \, s_n \qquad \text{(sum runs over nearest-neighbor pairs)}$$

Energy   Coupling constant n-th spin, +1 or −1

The formula gives the energy of the system as a function of its *micro-state*, which is the orientation of all the spins. The 2D Ising model is famous because it can be solved analytically.

Boltzmann Probability Distribution.

The *Boltzmann probability distribution* says that the probability of finding a system in a given micro-state depends exponentially on the energy of the micro-state, and the temperature of the system:

$$P(s) \propto e^{-\beta E(s)} \qquad \beta \propto \frac{1}{T}$$

Probability of state

Energy of state

Temperature

Expressed in words, the Boltzmann distribution says two things.

- Physical systems tend to assume lower-energy states.
- If you pump a certain amount of energy into a system (for example by heating it), than all accessible states with that amount of energy or less are about equally likely.

Equipartition Theorem.

Equipartition Theorem: Suppose that (1) a system can be decomposed into a set of degrees of freedom, and (2) the total energy of the system is the sum of the energies of the individual degrees of freedom. Then the system behaves as if the degrees of freedom are independent random variables, with the Boltzmann distribution applying separately to each degree of freedom.

Suppose further that (3) the energy of an individual degree of freedom depends quadratically on its value. Then each degree of freedom has the same expected energy:

$$\langle E_n \rangle = \frac{1}{2\beta} \qquad \text{for all } n$$

Expected energy of n-th degree of freedom

Example: An ideal gas, where the degrees of freedom are the momentum components of the molecules. Each momentum component p_n has associated energy $E_n = p_n^2/2m_n$ which is a quadratic function of p_n. The total energy of the ideal gas is $\sum E_n$.

Thermodynamic Fault Model

Steps for constructing a heterogeneous stress pattern by analogy to the Ising model.

1. The system is the earthquake fault and surrounding rock.

2. A micro-state is defined to be a pattern of slip on the fault.

3. The energy E of a given micro-state is defined to be the elastic energy associated with the given pattern of slip.

4. Choose a thermodynamic parameter β. (This will turn out to be equivalent to choosing a correlation length.)

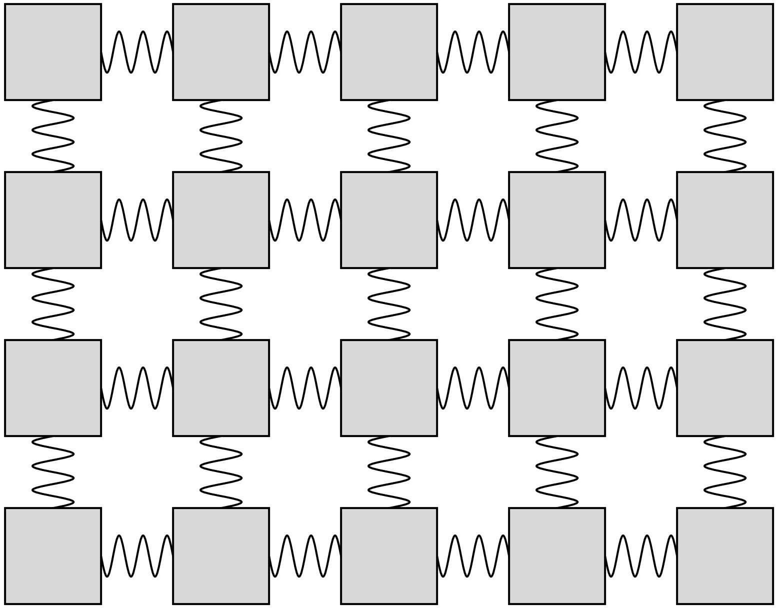
5. Use a computer to randomly construct a micro-state, so that the probability of choosing a given pattern of slip equals its Boltzmann probability (which depends on its energy).

6. To make the computation feasible, decompose the system into a set of degrees of freedom. Then, by equipartition, the state of each degree of freedom can be selected independently.

7. Compute the pattern of shear stress resulting from the selected pattern of slip.

The rest of this talk presents the machinery for carrying out this program.

Wrong Approach — Sliders and Springs.



This is the WRONG way to apply the Ising model.

The figure is a slider-and-spring model of an earthquake fault. Each rectangular block can slide within the plane, and is connected by springs to its four nearest neighbors.

Because this figure looks so much like the Ising model, there is a temptation to say that it is the analog of the Ising model. But the slider-and-spring model is fatally flawed, because it has the wrong energy spectrum.

Energy Spectrum for Sliders and Springs.

Let's look at the energy spectrum of the slider-and-spring model. Suppose we impose a sinusoidal displacement on the blocks:

$$u(x_0, x_1) = F \cos(k_0 x_0 + k_1 x_1)$$

Displacement of block at (x_0, x_1) Amplitude   Wavenumbers

Then the energy stored in the springs is:

$$E_{\text{spring}} \propto F^2 k^2 \qquad \text{where} \quad k^2 = k_0^2 + k_1^2$$

But if you impose a sinusoidal displacement on the surface of a 3D elastic medium, the energy stored in the elastic medium is

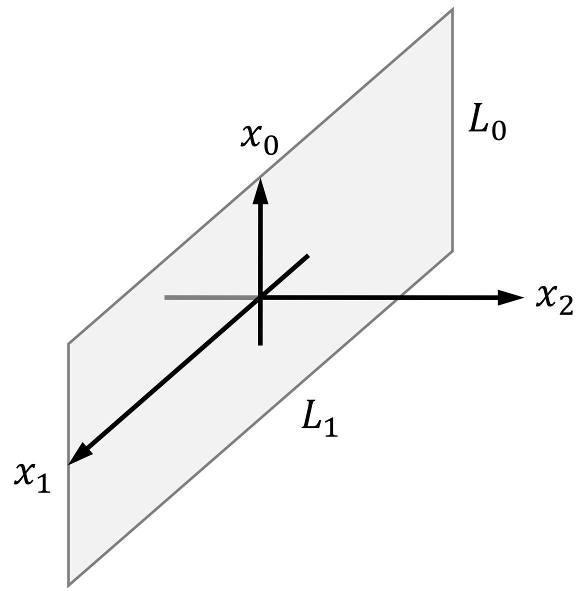
$$E_{\text{elastic}} \propto F^2 k$$

The difference arises because, for the 3D elastic medium, long-wavelength displacements penetrate more deeply into the medium. So long-wavelength (low k) modes have more energy compared to the slider-and-spring case, because they distort a larger volume of medium.

The Boltzmann distribution says that the probability of a state depends on its energy. So, if the energy is wrong, then the Boltzmann distribution cannot give useful results.

Right Approach — 3D Elastic Volume.

So, we need to work in a 3D elastic volume. Interactions in a 3D medium are not limited to "nearest neighbors" and so we need a formulation that allows long-range interactions.



We define the three coordinate axes as follows:

- The x_0 axis runs along-dip.
- The x_1 axis runs along-strike.
- The x_2 axis is perpendicular to the fault.

We envision the fault as being strike-slip. (For dip-slip, rotate everything by 90 degrees.)

We impose periodic boundary conditions in the x_0 and x_1 directions (but not the x_2 direction), with lengths L_0 and L_1respectively, so the fault area is $L_0 L_1$. Our calculations implicitly assume a limit as the periodic boundary conditions go to infinity; $L_0, L_1 \to \infty$.

Orthogonal Set of Displacement Fields.

Introduce the following displacement fields in the 3D elastic volume. They form a complete and orthogonal set of approximate solutions to the elastostatic fault-slip problem.

$$u_0(x_0, x_1, x_2) = -F(k_0, k_1)\frac{k_0 k_1}{k^2 + k_0^2}\; e^{-k|x_2|} \cos\big(k_0 x_0 + k_1 x_1 + \Phi(k_0, k_1)\big)\, \mathrm{sgn}(\mathrm{x}_2)$$

$$u_1(x_0, x_1, x_2) = F(k_0, k_1)\; e^{-k|x_2|} \cos\big(k_0 x_0 + k_1 x_1 + \Phi(k_0, k_1)\big)\, \mathrm{sgn}(x_2)$$

$$u_2(x_0, x_1, x_2) = -F(k_0, k_1)\frac{k k_1}{k^2 + k_0^2}\; e^{-k|x_2|} \sin(k_0 x_0 + k_1 x_1 + \Phi(k_0, k_1))$$

Displacement at (x_0, x_1, x_2) Amplitude Wavenumbers Phase

The scalar wavenumber is:

$$k = \sqrt{k_0^2 + k_1^2}$$

Basic Properties of the Displacement Fields.

1. Away from the fault, the displacements are an exact solution of the elastostatic equation:

$$\sigma_{ij,j} = 0 \qquad \sigma_{ij} = \lambda\delta_{ij}\varepsilon_{ll} + 2\mu\varepsilon_{ij} \qquad \varepsilon_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j})$$

Stress tensor   Shear modulus   Strain tensor   Displacement field

2. The shear stress τ is the same on both sides of the fault, as it should be.

3. The normal stress σ is **not** the same on both sides of the fault. *This is what makes the displacement field an **approximate** solution to the elastostatic fault slip problem*.

4. At the fault, the displacements describe fault slip which varies sinusoidally, with wavenumbers k_0 and k_1, and with amplitude proportional to $F = F(k_0, k_1)$.

5. Away from the fault, the displacements die off exponentially over characteristic distance $1/k$.

This shows that fault slip with long wavelength (small k) penetrates more deeply into the 3D medium than fault slip with short wavelength (large k).

Energy of the Displacement Fields.

6. The elastic energy in the 3D medium is:

$$E(k_0, k_1) = \frac{1}{2}\,\mu\, L_0 L_1 \left(\frac{2k^2}{k^2 + k_0^2}\right)^2 k\, F(k_0, k_1)^2$$

Elastic energy Shear modulus Fault area Wavenumber Amplitude squared

This is the expected energy spectrum, with $E \propto k\, F^2$.

7. The displacement fields are *orthogonal*. If you form a linear combination of these displacement fields, with different wave vectors and phases, then the total elastic energy of the linear combination equals the sum of the energies of the individual fields:

$$E_{\text{total}} = \frac{L_0 L_1}{4\pi^2} \int E(k_0, k_1)\, dk_0\, dk_1$$

Total elastic energy Elastic energy contributed by wave vector (k_0, k_1)

Orthogonality makes our system satisfy the conditions of the equipartition theorem. We want to randomly generate a state of the system, so that the total energy of the system obeys the Boltzmann distribution. According to the equipartition theorem, we can do this by selecting each amplitude $F(k_0, k_1)$ *independently* according to the Boltzmann distribution. This makes our method computationally feasible.

Shear Stress of the Displacement Fields.

8. The shear stress on the fault is parallel to the x_1 axis, that is, it is oriented along-strike. It is:

$$\tau(k_0, k_1) = -\sqrt{\frac{2\mu E(k_0, k_1)k}{L_0 L_1}} \cos\big(k_0 x_0 + k_1 x_1 + \Phi(k_0, k_1)\big) \operatorname{sgn}(F(k_0, k_1))$$

9. The mean-square shear stress for a given wave vector and phase (or *power spectral density*) is:

$$\tau_{\text{rms}}^2(k_0, k_1) = \frac{\mu}{L_0 L_1} E(k_0, k_1)\, k$$

10. For a linear combination of displacement fields, the total mean-square shear stress is the sum of the mean-square stresses of the individual fields (Parseval's Theorem):

$$\tau_{\text{total}}^2 = \frac{L_0 L_1}{4\pi^2} \int \tau_{\text{rms}}^2(k_0, k_1)\, dk_0\, dk_1 = \frac{\mu}{4\pi^2} \int E(k_0, k_1)\, k\, dk_0\, dk_1$$

Now recall the second part of the equipartition theorem: the expected energy $\langle E(k_0, k_1)\rangle$ is equal to $1/2\beta$ for all wave vectors and phases. Then the expected shear stress $\langle \tau_{\text{rms}}^2(k_0, k_1)\rangle$ is *isotropic* because its value does not depend on the direction of the wave vector, only its magnitude. So our thermodynamic method predicts that the shear stress pattern is isotropic.

But the formula for τ_{total}^2 shows we have a problem: the integral diverges. **So the Boltzmann distribution predicts an infinite amount of stress.** This is called an *ultraviolet divergence* because the integral diverges in the limit of large k, or small wavelength.

Ultraviolet Divergence

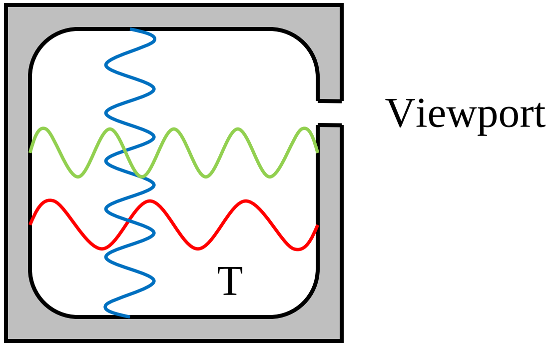
The ultraviolet divergence is not a new problem. In some prior methods, the power spectral density behaves asymptotically like a power law, in the limit of large k:

$$\tau_{\mathrm{rms}}^2(k_0, k_1) \sim \frac{1}{k^\alpha} \qquad \text{as } k \to \infty$$

Any such method has an ultraviolet divergence if $\alpha \leq 2$, because in that case $1/k^\alpha$ falls off so slowly that it is not integrable over the $k_0 k_1$-plane.

This is true for the 100 Runs method, and for the Andrews & Barall method, which both use $\alpha = 2$.

Blackbody Radiation Problem.



The term *ultraviolet divergence* gets its name from a famous problem in statistical mechanics: **computing the spectrum of blackbody radiation**.

The figure shows a container at a temperature T. There is electromagnetic radiation inside the container, and a small viewport lets the experimenter observe it. The problem is to calculate how much radiation exists at each frequency.

The equipartition theorem says each wave vector and polarization has the same expected energy $1/2\beta$.

But there are an infinite number of wave vectors in the ultraviolet (and higher) frequency range. So the Boltzmann distribution predicts that the container holds an infinite amount of ultraviolet energy. This does not agree with observation.

Planck's Resolution of Blackbody Problem.

For blackbody radiation, the ultraviolet divergence was solved by Planck about 100 years ago.

The key idea is to restrict the set of accessible states of the system. The Boltzmann distribution is then applied to the new, smaller set of states.

(Recall that the Boltzmann distribution says the probability of a state s is $P(s) \propto e^{-\beta E(s)}$. We still use the same Boltzmann distribution, only now s runs over a smaller set.)

Originally, the amplitudes were allowed to be arbitrary real numbers. Planck said that, instead, the amplitudes are only allowed to assume certain discrete values, called *quantum levels*. By doing this, Planck reduced the number of states that the system can assume.

This change reduces the expected energy of the high-k modes to below $1/2\beta$, which makes the total energy finite.

Finite Fault Strength Restricts the Set of Accessible States.

We seek to solve our ultraviolet divergence in a manner analogous to what Planck did — we want to restrict the set of accessible states so that the total shear stress becomes finite. The question is: for earthquake faults, what physical process restricts the set of possible states? Answer:

The finite strength of the earthquake fault acts to restrict the set of accessible states.

We originally allowed our displacement amplitudes to be arbitrary real numbers. But this means that many states have shear stress which exceeds the strength of the fault. Such states cannot occur, because an earthquake would happen before such a state is attained.

So, we should not be applying the Boltzmann distribution to the set of all possible linear combinations of displacement fields. Instead, we should apply it only to linear combinations that produce shear stress that the fault can sustain.

In order to obtain a mathematically tractable method, **we will carry out this restriction by limiting the maximum expected mean-square stress** (rather than attempting to force the shear stress to be below the yield stress at every single point on the fault).

Qualitative Discussion

Recall the equation that relates expected mean-square shear stress to expected energy:

$$\langle \tau_{\text{total}}^2 \rangle = \frac{\mu}{4\pi^2} \int \langle E(k_0, k_1) \rangle \, k \, dk_0 \, dk_1$$

Expected total mean-square shear stress

Expected energy for wave vector (k_0, k_1)

Now we treat the expected mean-square stress $\langle \tau_{\text{total}}^2 \rangle$ as being fixed by the strength of the fault. We explore, qualitatively, the consequences for the expected energy $\langle E(k_0, k_1) \rangle$.

- For very small k, the factor of k in the integrand means that $\langle E(k_0, k_1) \rangle$ contributes very little to the stress. In other words, small-k modes do not “feel” the finite strength of the fault very much. So we expect that $\langle E(k_0, k_1) \rangle$ remains approximately equal to $1/2\beta$.

- For very large k, the factor of k in the integrand means that $\langle E(k_0, k_1) \rangle$ contributes a lot to the stress. In other words, large-k modes do “feel” the finite strength significantly. So we expect that $\langle E(k_0, k_1) \rangle$ is substantially less than $1/2\beta$.

- As $k \to \infty$, the expected energy $\langle E(k_0, k_1) \rangle$ must approach zero quickly enough so the integral remains finite.

Energy Rolloff Function and Correlation Length.

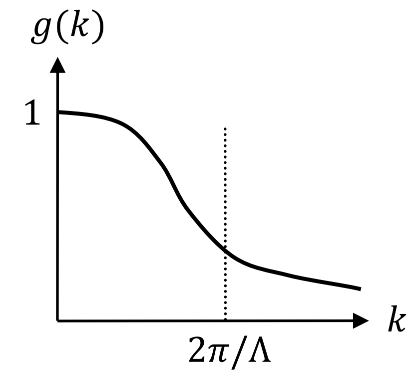
Let's express these consequences by introducing a function $g(k)$ defined by

$$\langle E(k_0, k_1) \rangle = \frac{1}{2\beta}\; g(k)$$

$$g(0) = 1$$

$$0 < g(k) \leq 1$$

$$\langle \tau_{\text{total}}^2 \rangle = \frac{\mu}{4\pi\beta} \int g(k)\, k^2\, dk < \infty$$



The expected energy $\langle E(k_0, k_1) \rangle$ equals $1/2\beta$ when $k \approx 0$, but approaches zero when $k \to \infty$.

Up until now, everything we did was scale-invariant, that is, there was no preferred length scale. But now, in order for the above integral to be finite, $g(k)$ has to approach zero over some characteristic length Λ. We call this the *correlation length*. So:

If the fault has finite strength, then there must be a length scale in the system, which is the correlation length.

The last equation implies a relation between β and the correlation length:

$$\beta \propto \Lambda^{-3}$$

Gutenberg-Richter and Self-Similarity

Earthquake Rates, Scaling, and Self-Similarity.

Some authors (including me) have said that earthquake self-similarity implies that the stress distribution is self-similar, which implies that its power spectral density is $1/k^2$. We will show that the correct formula is $1/k^2(\log(\zeta k))^2$ where ζ is a length scale.

Following Hanks (1977 & 1979) **we make the following assumptions about earthquakes**.

Assumption 1. Gutenberg-Richter law:

$$\log N = a - M$$

Rate of earthquakes of magnitude $\geq M$

Magnitude

Assumption 2. Magnitude-moment scaling relation:

$$\log M_0 = 1.5M + d \qquad M_0 \propto \delta\, r^2$$

Seismic moment

Slip

Radius of earthquake

Assumption 3. Self-similarity or scale-invariance:

$$\delta \propto r \qquad \Delta\tau \propto r^0$$

Stress drop is independent of r

Earthquake Rate ≠ Stress Perturbation Frequency.

A little algebra shows that the rate of earthquakes with radius between r and $r + dr$ is:

$$N_{\mathrm{GR}}(r)\,dr = -\frac{dN}{dr}\,dr \;\propto\; r^{-3}\,dr$$

Hanks offers the following interpretation of the above formula. Imagine a very large fault, with random stress perturbations of various sizes. Each positive stress perturbation is an "incipient earthquake." These incipient earthquakes follow Gutenberg-Richter statistics, that is, $N_{\mathrm{GR}}(r)\,dr$ is proportional to the number of stress perturbations of radius between r and $r + dr$.

But, try to calculate the total area of all stress perturbations with radius $\leq R$:

$$A(r \leq R) \;\propto\; \int_0^R N_{\mathrm{GR}}(r)\,r^2\,dr \;\propto\; \int_0^R r^{-1}\,dr = \;\infty$$

The integral diverges. **There isn't enough room on the fault to hold all the small stress perturbations if they follow Gutenberg-Richter statistics.** This is another example of an *ultraviolet divergence* because the divergence occurs at small scales.

The problem occurs because Gutenberg-Richter gives the rate at which earthquakes occur over an *interval* of time, while stress perturbations exist simultaneously at an *instant* of time, so it's incorrect to equate the two.

Earthquake Repetition, and the Resolution of the Ultraviolet Divergence.

To find the correct statistics, we must take account of the fact that earthquakes repeat, and smaller earthquakes repeat more frequently. Because earthquakes of different sizes repeat at different rates, the stress perturbations existing at an instant of time will have statistics that differ from Gutenberg-Richter.

We introduce the following additional assumption.

Assumption 4. Constant rate of slip deficit accumulation:

$$T \propto \delta$$

Time for earthquake to repeat   Slip

Now the number of stress perturbations or "incipient earthquakes" with radius between r and $r + dr$ is:

$$N_{\mathrm{SP}}(r)\, dr = T\, N_{\mathrm{GR}}(r)\, dr \propto r^{-2}\, dr$$

Then the total area of all stress perturbations with radius $\leq R$ is:

$$A(r \leq R) \propto \int_0^R N_{\mathrm{SP}}(r)\, r^2\, dr \propto \int_0^R dr = R$$

This resolves the ultraviolet divergence.

The Implied Stress Power Spectral Density.

The next step is to calculate the implied power spectral density of the stress field. We just showed that the total area of stress perturbations or "incipient earthquakes" with radius $\leq 1/k$ is:

$$A(r \leq 1/k) \propto 1/k$$

According to our formulas, the expected mean-square stress due to wavenumbers $\geq k$ is proportional to:

$$\langle \tau_{\text{total}}^2 (k' \geq k) \rangle = \int_k^\infty \langle \tau_{\text{rms}}^2 (k_0', k_1') \rangle \, k' \, dk' = \frac{\mu}{4\pi\beta} \int_k^\infty g(k') \, k'^2 \, dk'$$

Expected power spectral density

Energy roll-off function

We now ask the question: to obtain $A \propto 1/k$ in the limit of large k, what must be the asymptotic behavior of the power spectral density and energy roll-off? This is a lengthy calculation. If we assume a Gaussian distribution of stress perturbations, then the answer is:

$$\langle \tau_{\text{rms}}^2 (k_0, k_1) \rangle \propto \frac{1}{k^2 (\log(\zeta k))^2} \qquad \text{as } k \to \infty$$

$$g(k) \propto \frac{1}{k^3 (\log(\zeta k))^2} \qquad \text{as } k \to \infty$$

The parameter ζ has dimensions of length and is a length scale related to the correlation length.

Stress Generation Algorithm

Energy Rolloff Function and State Restriction.

Recall that we are imposing a restriction on expected energy:

$$\langle E(k_0, k_1) \rangle = \frac{1}{2\beta}\, g(k)$$

We know that $g(0) = 1$, and in the previous section we showed that

$$g(k) \propto \frac{1}{k^3(\log(\zeta k))^2} \qquad \text{as } k \to \infty$$

We need to choose a functional form for $g(k)$ which has the above asymptotic behavior. This could be done in any number of ways, but we have chosen the following:

$$g(k) = \frac{1}{1 + (\zeta k)^3(\log(1 + \zeta k))^2}$$

In order to obtain the desired energy, we impose a constraint on the maximum amplitude:

$$|F(k_0, k_1)| \leq R(k_0, k_1)$$

The constraint value $R(k_0, k_1)$ is computed so that, when the amplitude $F(k_0, k_1)$ is chosen according to the Boltzmann distribution, and in accordance with the above constraint, the expected value of the energy is $g(k)/2\beta$.

Algorithm for Generating Random Stresses.

Step 1. The only parameter we need to specify is the correlation length Λ. Once we know Λ, we can obtain the energy roll-off function $g(k)$.

Step 2. For each wave vector, we independently select amplitude $F(k_0, k_1)$ and phase $\Phi(k_0, k_1)$ according to the Boltzmann distribution, satisfying the constraint $|F(k_0, k_1)| \leq R(k_0, k_1)$.

Step 3. The final random stress field is obtained by summing the stress fields of all the wave vectors. This can be done with a Fourier transform.

Selecting Rupture Extent and Correlation Length

In the 100 Runs project, we limited the rupture extent in a very simple way: we placed hard barriers at the edges of a pre-defined rectangular rupture area.

This time, **we want the rupture to be limited by the stress distribution — the rupture stops when it reaches an area of low stress**. This has two advantages:

- It allows the size and shape of the rupture area to vary.

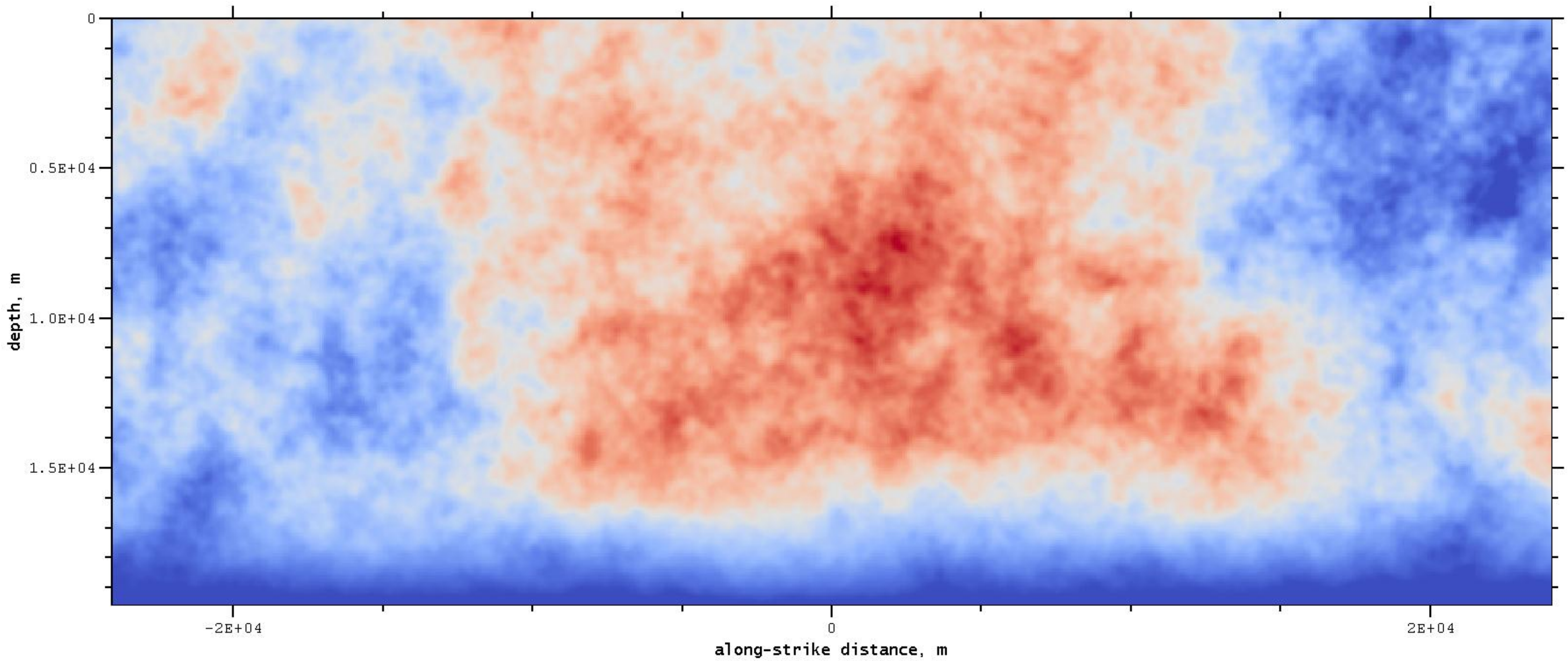
- It avoids the strong seismic wave reflections that occur when the rupture reaches a hard barrier.

Andrews & Barall (2011) presented one way to do this, by manipulating the Fourier transform of the stress field. Here we present a new and better technique, which has the following advantages:

- Does not require manipulation of the randomly-generated stress field.

- Provides a way to determine the correlation length.

- Produces a heavy-tailed stress distribution, which survives low-pass filtering.



Here is an example of a stress distribution produced by our method, with red showing areas of positive stress drop.

Rupture extent is limited at the bottom and the sides by the blue areas of negative stress drop.

The blue area at the bottom is created by *depth conditioning*. The blue areas at the sides are created by the random stress distribution, together with a *selection technique*.

Depth Conditioning.

Rupture extent along-dip is controlled by *depth conditioning*, a post-processing step that systematically reduces the shear stress with increasing depth. In the example shown, the shear stress is modified by multiplying it with a depth-conditioning function:

$$\tau(x_0, x_1) \leftarrow \tau(x_0, x_1) \cdot \eta(x_0)$$

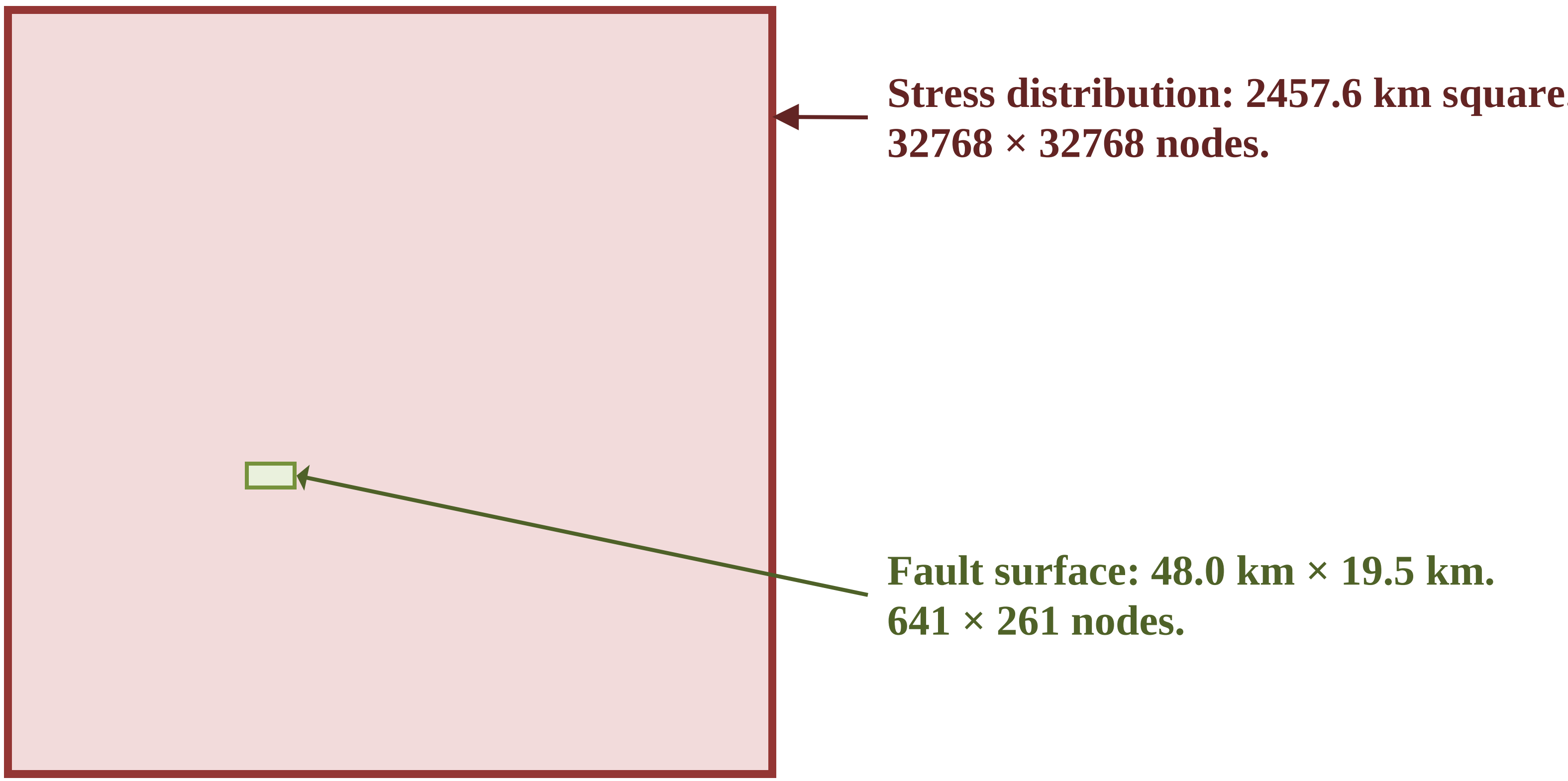
We take the depth-conditioning function $\eta(z)$ to be a piecewise-linear function, linearly interpolated between the following data points.

Vertical coordinate x_0 in meters	Depth-conditioning function $\eta(x_0)$
0	1.00
−14000	1.00
−16000	0.80
−19500	0.10

In this example, the depth-conditioning function equals 1.00 in the uppermost 14 km of the fault surface.

Depth conditioning is justified by the fact that the properties of the fault change systematically at the base of the seismogenic zone.

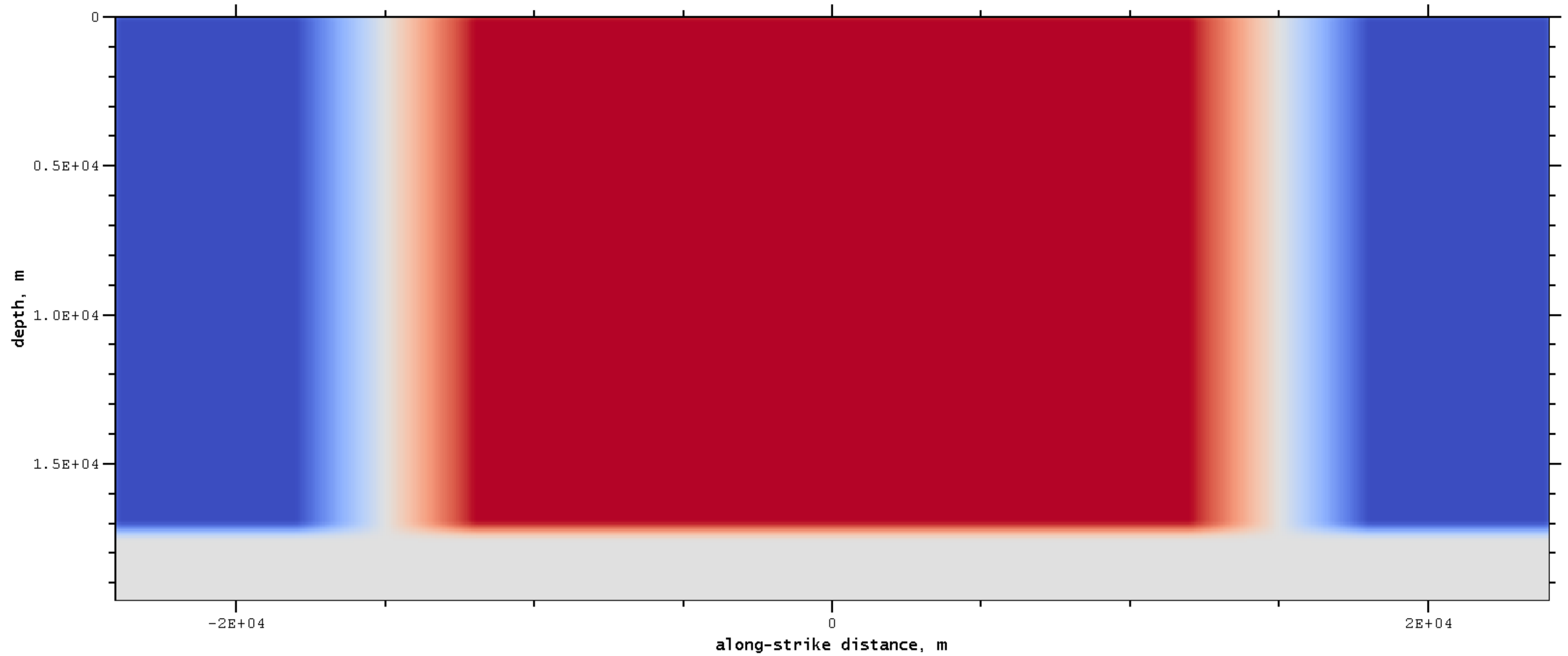
Selection Process.



Rupture extent along-strike is controlled by a *selection process*, which selects the fault surface from a stress distribution thousands of times larger.

The idea is to generate a random stress distribution over a very large area. Then, select a fault surface where the randomly-generated stress just happens to have the desired pattern, which is high in the center of the fault and low at the edges.

Selection Algorithm



Step 1. Create a *selection template*. It is a function $Q(x_0, x_1)$ which is positive (red) in the portion of the fault surface where we hope to have high stress, and negative (blue) in the portion of the fault surface where we hope to have low stress. **Our objective is to find the place in the large randomly-generated stress distribution which is most similar to the selection template.**

The selection template is adjusted to have zero DC component, so it is sensitive to variations in stress rather than the absolute level of stress.

In the example shown, the function is zero (gray) at the bottom of the fault, because depth conditioning forces the stress to be low in that area.

Selection Algorithm (continued)

Step 2. Find the point $(\tilde{x}_0, \tilde{x}_1)$ which maximizes the value of the integral

$$\int Q(y_0, y_1)\, \tau(\tilde{x}_0 + y_0, \tilde{x}_1 + y_1)\, dy_0\, dy_1$$

Selection template Q Randomly-generated shear stress τ

The point $(\tilde{x}_0, \tilde{x}_1)$ can be located using Fourier transforms, because the integral is a convolution.

Step 3. Place the upper left corner of the fault surface at location $(\tilde{x}_0, \tilde{x}_1)$. Cut out the part of the stress distribution $\tau(x_0, x_1)$ that lies in the fault surface, and throw away the rest.

Correlation Length.

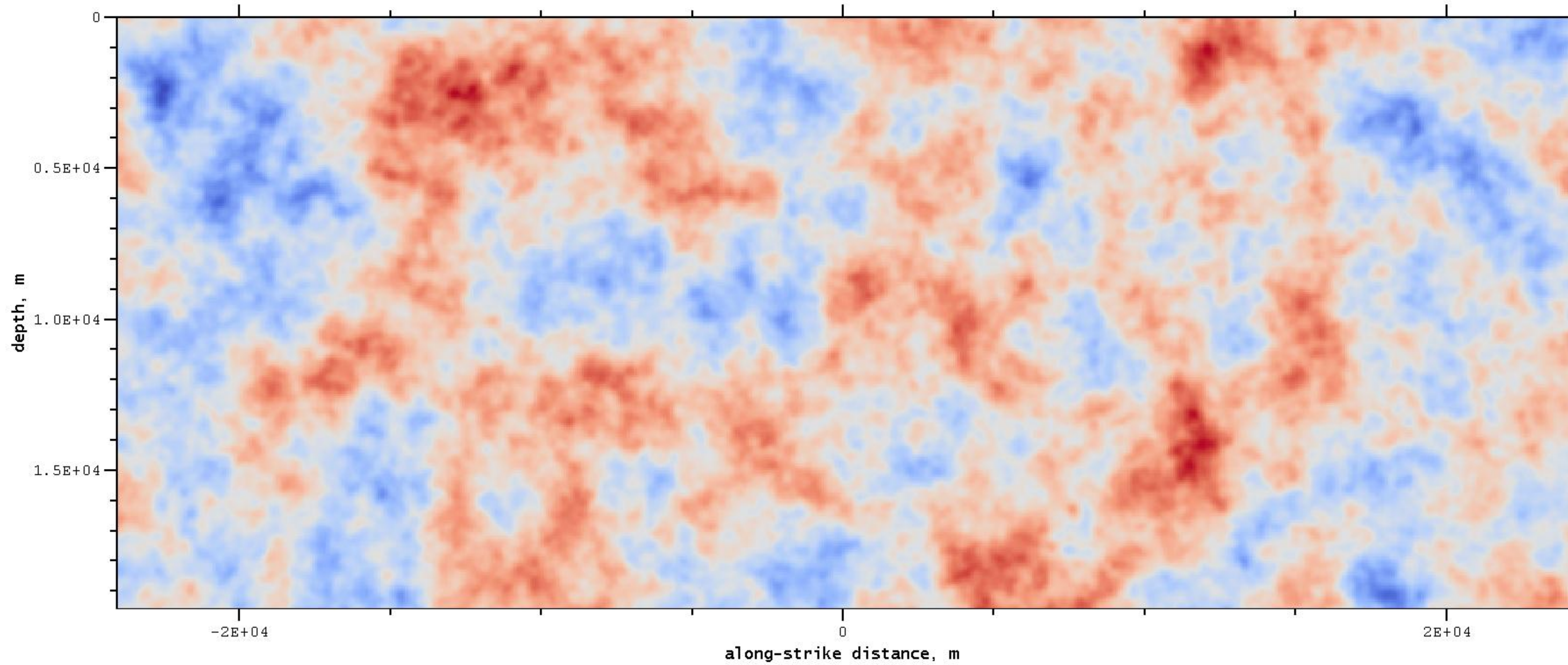
The selection process also gives us a way to determine the correlation length, although it requires some trial-and-error.

If the correlation length is set too small, then it will be impossible to find a stress pattern that supplies high stress over the desired area of the rupture. As a result, it won't be possible to successfully execute a dynamic rupture simulation.

Our recipe is: **Select the smallest correlation length that reliably gives stress patterns that can sustain a rupture of the desired size**.

In general, it appears that the correlation length should be about equal to the desired diameter of the rupture.

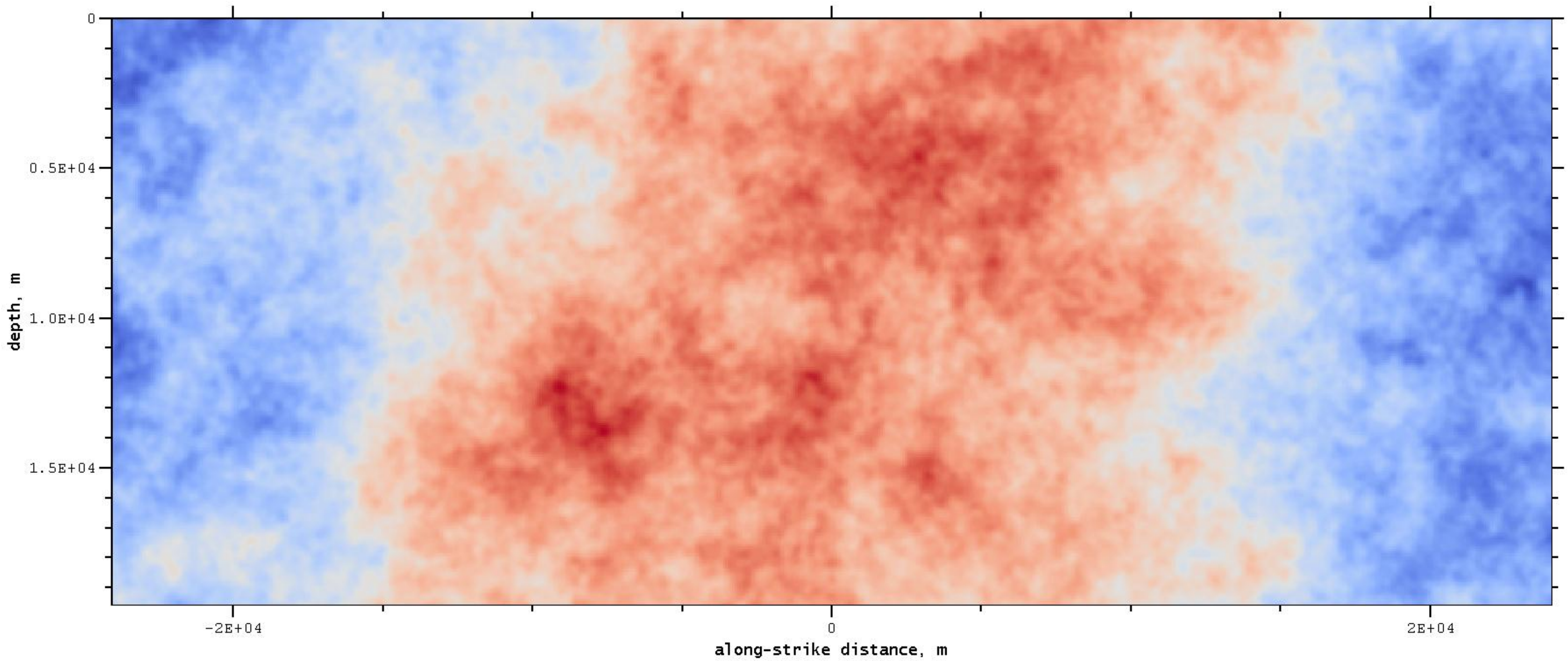
As illustration, the following three slides show stress patterns from correlation lengths that are too small, too large, and just right.



Correlation length = 10 km. This is too small.

We are trying to have a rupture 30 km across, but with this small correlation length it is impossible to find any stress perturbations that are large enough.

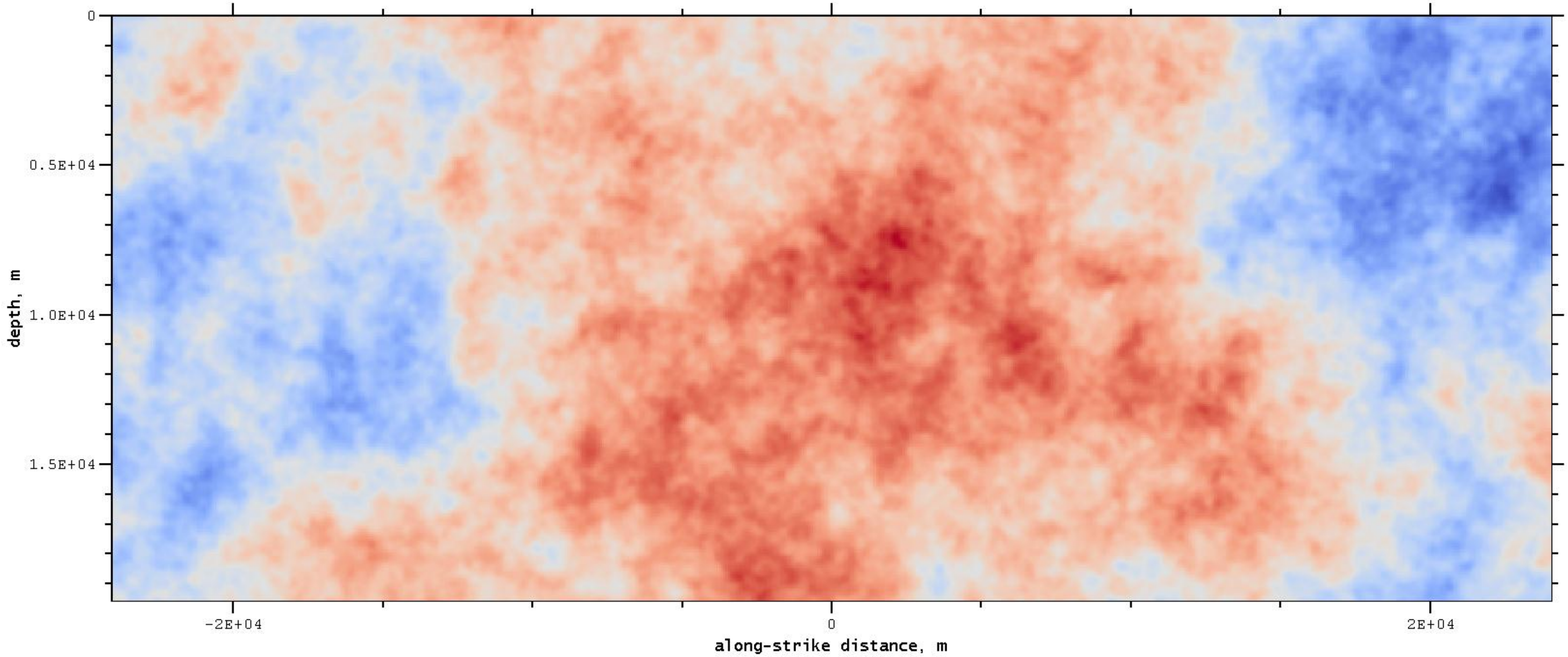
We just have intermixed high-stress and low-stress areas which cannot sustain a rupture of the desired size.



Correlation length = 100 km. This is too large.

We see a very clean separation of high-stress (red) and low-stress (blue) areas.

This stress pattern would sustain a rupture, but the separation is too good.



Correlation length = 30 km. This is just right.

The border between high-stress (red) and low-stress (blue) areas is ragged. There are pockets of high stress within the low-stress area and vice-versa.

The separation between high-stress and low-stress is just good enough to sustain a rupture, that stops at about 30 km diameter.

Post-Processing — Filtering and Scaling.

After generating the random stresses, we need to perform some post-processing steps.

1. Apply a low-pass spatial filter, to suppress details too small to be resolved by the dynamic rupture code (*e.g.*, smaller than 3 to 5 cells).

2. Scale the shear stress to be consistent with the friction law.

3. If the initial normal stress is variable, then scale the shear stress by a factor proportional to the normal stress (*i.e.*, treat our stress values as lying in “friction coefficient space”).

One-Point Statistics.

Given a stress pattern, its *one-point statistics* are obtained by treating each stress value as an instance of a random variable, and then analyzing the statistical properties of that random variable.

For the present discussion, we distinguish only between *Gaussian* statistics and *heavy-tailed* statistics. We consider the statistics to be heavy-tailed if the tails of the probability distribution fall off distinctly more slowly than one would expect of a Gaussian. Using a simple test left over from the 100 Runs project, I find:

- For the entire large stress distribution, the one-point statistics are Gaussian.

- **For the fault surface, the one-point statistics are heavy-tailed. This remains true regardless of whether or not filtering is used.**

Conclusion

Conclusion

We showed how to construct a model of fault slip, which can be treated by analogy to thermodynamics. But it predicts infinite stress.

By considering earthquake scaling laws and self-similarity, we showed that the expected power spectrum should be like $1/k^2(\log(\zeta k))^2$ for large k.

Then we can resolve the ultraviolet divergence, and use the Boltzmann distribution to generate random stress patterns.

We showed how to limit the rupture extent, and find a correlation length, by selecting the fault surface from a much larger stress pattern.

Thank You