Finite Difference Quake and Wave Laboratory (FD-Q-WaveLab)

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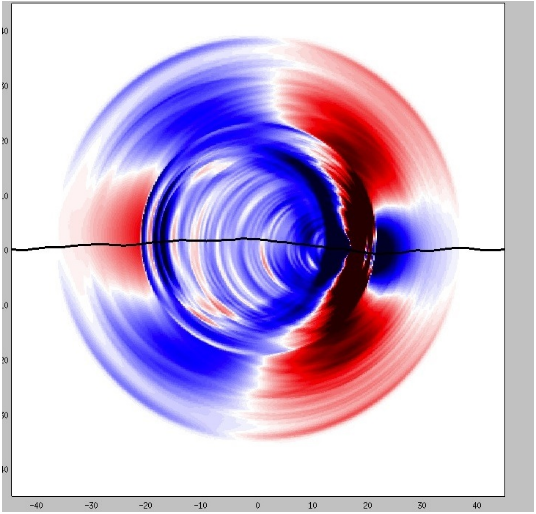
Main features

- Heterogeneous media.
- Complex geometry.
- High order accurate.
- Explicit high order accurate time-stepping (Runge-Kutta 4).
- Provably stable.

Similar approach



FDMap (2D finite difference): J. Kozdon + E. Dunham.

Elastic wave equation in 3D

Displacement formulation (u_i) : [Most finite element schemes]

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j},$$

Hooke's Law: $\sigma_{ij} = C_{ijkl}\epsilon_{ij}, \quad \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$

Isotropic elasticity: $C_{ijkl} = C_{ijkl}(\lambda, \mu)$.

Velocity–Stress formulation (v_i, σ_{ij}) : [Most finite diff. schemes]

$$\rho \frac{\partial v_i}{\partial t} = \frac{\partial \sigma_{ij}}{\partial x_j},$$

$$\frac{\partial \sigma_{ij}}{\partial t} = \lambda \delta_{ij} \frac{\partial v_k}{\partial x_k} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

Velocity field: v_i , Stress tensor: σ_{ij}.

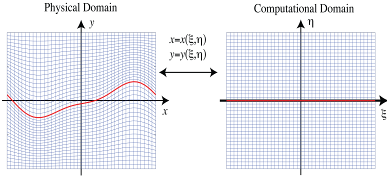
Friction law:

$$\tau = F(\sigma, v, \psi), \quad \frac{\partial \psi}{\partial t} = -G(\tau, v, \psi).$$

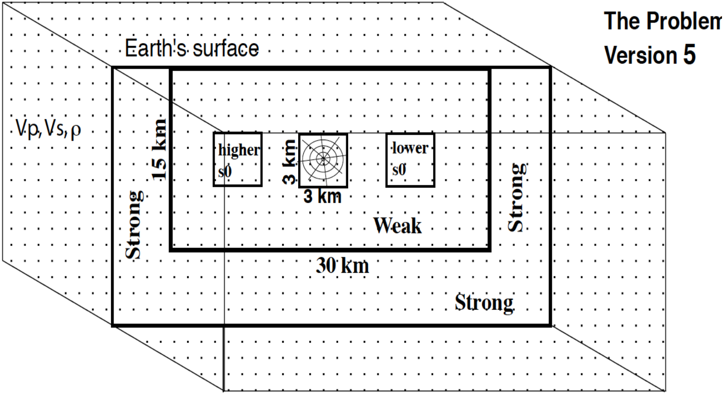
- τ is the shear stress on the fault.
- σ is compressive normal stress.
- $v = [\![\frac{\partial u}{\partial t}]\!]$ is the slip rate.
- ψ is a state variable (or slip).

What makes our code unique ?

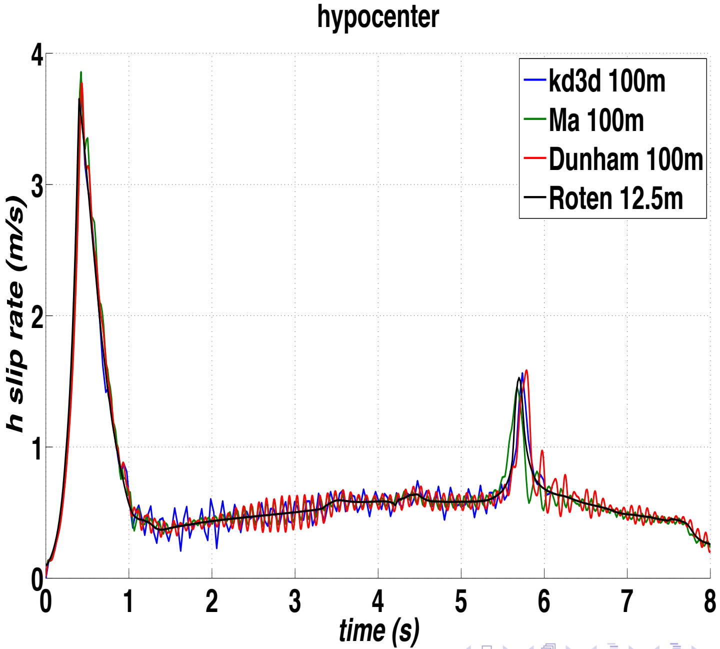
- Velocity–Stress formulation
- High order accuracy in the vicinity of a complicated fault geometry and for wave propagation
- All variables are collocated at the same grid points in both time and space
  - Efficient enforcement of fault and boundary condition
  - Numerical treatment of plasticity is less cumbersome
- Resolve faults geometry (Curvilinear grids)



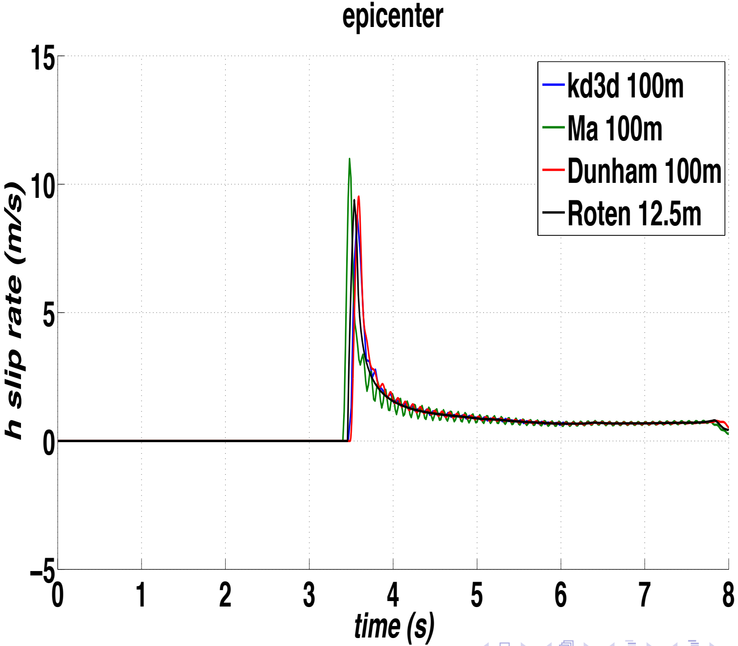
TPV5


The Problem
Version 5
Earth's surface
Vp, Vs, ρ
15 km
higher s0
3 km
3 km
lower s0
Strong
Strong
Weak
30 km
Strong

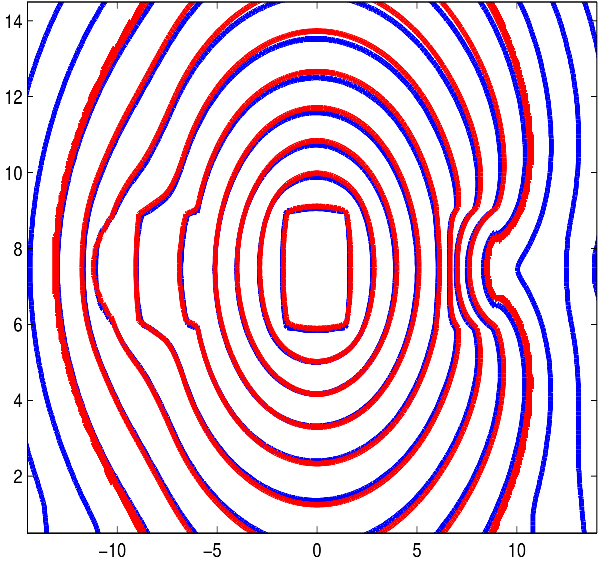
Hypocenter


hypocenter
kd3d 100m
Ma 100m
Dunham 100m
Roten 12.5m
h slip rate (m/s)
time (s)
0
1
2
3
4
5
6
7
8

Epicenter

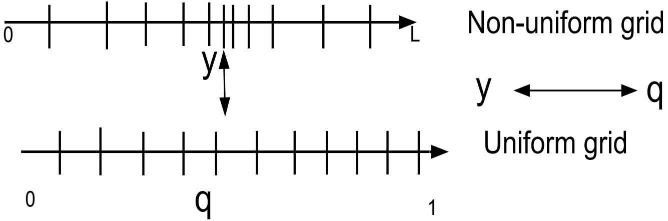

epicenter
15
10
5
0
−5
0
1
2
3
4
5
6
7
8
h slip rate (m/s)
time (s)
kd3d 100m
Ma 100m
Dunham 100m
Roten 12.5m

Rupture contours: KD vs Roten at h = 100m



How does our code work ?

Grid mapping



Non-conservative form : $$\frac{\partial v}{\partial y} = \frac{\partial q}{\partial y}\frac{\partial v}{\partial q} = q_y \frac{\partial v}{\partial q}$$

$$q_y := \frac{\partial q}{\partial y}, \quad J := y_q = \frac{\partial y}{\partial q}, \quad \frac{\partial q}{\partial y} = \left(\frac{\partial y}{\partial q}\right)^{-1} \Longleftrightarrow J = \frac{1}{q_y}.$$

Conservative form : $$\frac{\partial v}{\partial y} = \frac{1}{J}\frac{\partial}{\partial q}\left(J q_y v\right)$$

Numerical method

$$\rho \frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial y}, \quad \frac{1}{\mu}\frac{\partial \sigma}{\partial t} = \frac{\partial v}{\partial y}, \quad 0 \le y \le L.$$

$$\text{Boundary condition :} \quad \sigma = g(t) \equiv 0, \qquad y = 0.$$

Numerical method

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$$\rho \frac{\partial v}{\partial t} = \frac{1}{J}\frac{\partial}{\partial q}\left(J q_y \sigma\right), \quad \frac{1}{\mu}\frac{\partial \sigma}{\partial t} = q_y \frac{\partial v}{\partial q}.$$

$$\text{Energy :} \quad E_J(t) := \frac{1}{2}\int_0^1 \left(\rho v^2 + \mu^{-1}\sigma^2\right) J dq.$$

Energy rate:

$$\frac{d}{dt}E_J(t) = 0, \quad \text{for all} \quad t \ge 0.$$

Numerical stability

$$\text{Discrete energy :} \quad \mathcal{E}_{Jh}(t) := \frac{1}{2}\sum_{j=1}^{N}\left(\rho_j v_j^2 + \mu_j^{-1}\sigma_j^2\right) J_{jj} H_{jj}.$$

H_{jj} are quadrature weights given by the difference operator.

$$\text{Discrete energy rate :} \quad \frac{d}{dt}\mathcal{E}_{Jh}(t) = 0, \quad \text{for all} \quad t \geq 0.$$

Stable discrete approximation ?

Given: Wave equation + Initial Conditions + Boundary Conditions.

- Method of lines –discrete in space and continuous in time.
- Summation-By-Parts finite difference operators (in space).
- Weak enforcement of boundary conditions.
- Stability by energy methods (Stable systems of ODEs).
- Fully discrete: Explicit Runge-Kutta.

1D central differences

Grid points :

$$q_j = (j-1)\,h, \quad h = \frac{1}{N-1}, \quad j = 1, \cdots, N.$$

Grid function (Method of lines):

$$v_j(t) \approx v(q_j, t) \quad \mathbf{v}(t) = [v_1(t), v_2(t), v_3(t), \ldots, v_N(t)]^T.$$

Approximate spatial derivatives:

$$\frac{v_{j+1} - v_{j-1}}{2h} = \frac{\partial v(q_j)}{\partial q} + O\left(h^2\right), \quad j = 2, 3, \ldots N-1.$$

At $j = 1$ and $j = N$?

1D central differences

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At $j = 1$ and $j = N$?

$$\frac{v_2 - v_1}{h} = \frac{\partial v(q_1)}{\partial q} + O(h).$$

$$\frac{v_N - v_{N-1}}{h} = \frac{\partial v(q_N)}{\partial q} + O(h).$$

Differentiation matrix

$$\left(D\mathbf{v}\left(t\right)\right)_j \approx \frac{\partial v\left(q_j\right)}{\partial q}.$$

$$D = \frac{1}{h}\begin{bmatrix} -1 & 1 & 0 & & \\ -0.5 & 0 & 0.5 & & \\ & \ddots & \ddots & \ddots & \\ & & -0.5 & 0 & 0.5 \\ & & 0 & -1 & 1 \end{bmatrix}, \quad H = h\begin{bmatrix} 0.5 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0.5 \end{bmatrix}$$

Differentiation matrix

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Discrete:

$$\langle D\mathbf{v}, \mathbf{v}\rangle_H = -\langle \mathbf{v}, D\mathbf{v}\rangle_H + v_N^2 - v_0^2 \quad \{\text{Summation-By-Parts}\},$$

Continuous:

$$\left(\frac{\partial u}{\partial y}, u\right) = -\left(u, \frac{\partial u}{\partial y}\right) + u(1)^2 - u(0)^2 \quad \{\text{Integration-By-Parts}\}. \tag{1}$$

6th Order accuracy

$$(D\mathbf{v}(t))_j = -\frac{1}{60h}v_{j-3}+\frac{3}{20h}v_{j-2}-\frac{3}{4h}v_{j-1}+\frac{3}{4h}v_{j+1}-\frac{3}{20h}v_{j+2}+\frac{1}{60h}v_{j+3}$$

$j = 4, 5, \ldots, N-3$, What happens at $j = 1, 2, 3$?

6th Order accuracy

$$(D\mathbf{v}(t))_j = -\frac{1}{60h}v_{j-3} + \frac{3}{20h}v_{j-2} - \frac{3}{4h}v_{j-1} + \frac{3}{4h}v_{j+1} - \frac{3}{20h}v_{j+2} + \frac{1}{60h}v_{j+3}$$

$j = 4, 5, \ldots, N-3$, What happens at $j = 1, 2, 3$?

Modify $j = 1, 2, 3, 4, 5, 6$

$$\begin{bmatrix}
-\frac{21600}{13649} & \frac{104009}{54596} & \frac{30443}{81894} & -\frac{33311}{27298} & \frac{16863}{27298} & -\frac{15025}{163788} & & \\
-\frac{104009}{240260} & 0 & -\frac{311}{72078} & -\frac{20229}{24026} & -\frac{24337}{48052} & \frac{36661}{360390} & & \\
-\frac{30443}{162660} & \frac{311}{32532} & 0 & -\frac{11155}{16266} & \frac{41287}{32532} & -\frac{21999}{54220} & & \\
\frac{33311}{107180} & -\frac{20229}{21436} & \frac{485}{1398} & 0 & \frac{4147}{21436} & \frac{25427}{321540} & \frac{72}{5359} & \\
-\frac{16863}{78770} & -\frac{24337}{31508} & -\frac{41287}{47262} & -\frac{4147}{15754} & 0 & \frac{342523}{472620} & -\frac{1296}{7877} & \frac{144}{7877} \\
\frac{15025}{525612} & -\frac{3661}{262806} & \frac{21999}{87602} & -\frac{25427}{262806} & -\frac{342523}{525612} & 0 & \frac{32400}{43801} & -\frac{6480}{43801} & \cdots
\end{bmatrix}$$

Numerical method

Replace spatial derivatives with difference operators

$$\rho_j \frac{dv_j}{dt} = \frac{1}{J_j} D \left(J q_y \sigma \right)_j ,$$
$$\frac{1}{\mu_j} \frac{d\sigma_j}{dt} = q_{yj} (Dv)_j .$$

How do we impose boundary conditions ?

Numerical method

Replace spatial derivatives with difference operators

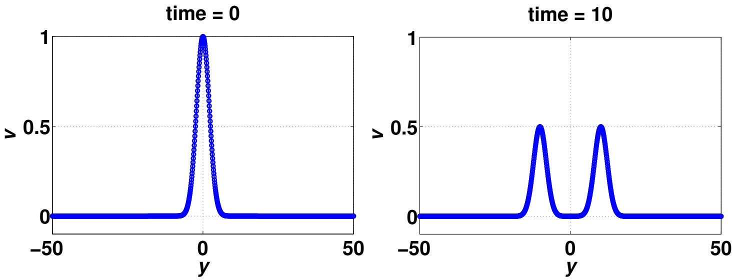
$$\rho_j \frac{dv_j}{dt} = \frac{1}{J_j} D \left(J q_y \sigma\right)_j \,,$$
$$\frac{1}{\mu_j} \frac{d\sigma_j}{dt} = q_{yj} (Dv)_j .$$

How do we impose boundary conditions ?

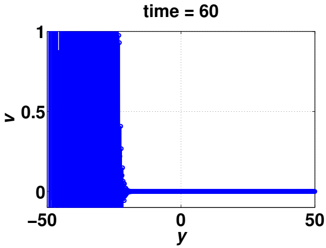
Strong enforcement: $\sigma_1 = g(t) \equiv 0$.

We can not prove stability!

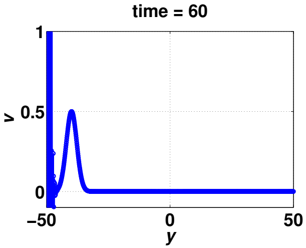
Initial data



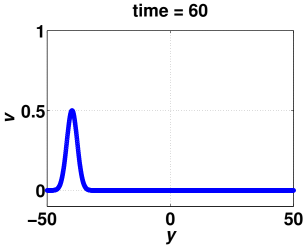
Strong BC: $\sigma_1 = g(t) \equiv 0$ (Unstable !)



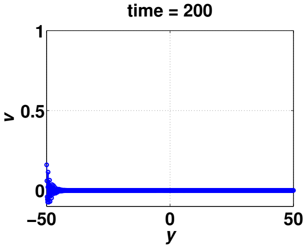
(a) sixth order accuracy



(b) fourth order accuracy



(c) second order accuracy



(d) second order accuracy

Weak boundary procedures

Simultaneous Approximation Term (SAT) —Carpenter et al. 1995

$$\text{Boundary:} \begin{cases} \rho_1 \frac{dv_1}{dt} = \frac{1}{J_1} D\left(Jq_y\sigma\right)_1 - \underbrace{\alpha H_{11}^{-1} q_{y1}\left(\sigma_1 - g(t)\right)}_{SAT}, \\ \frac{1}{\mu_1}\frac{d\sigma_1}{dt} = q_{y1}(Dv)_1 - \underbrace{\beta H_{11}^{-1} q_{y1}\left(\sigma_1 - g(t)\right)}_{SAT}. \end{cases} \tag{2}$$

$$\text{Interior:} \begin{cases} \rho_j \frac{dv_j}{dt} = \frac{1}{J_j} D\left(Jq_y\sigma\right)_j, \\ \frac{1}{\mu_j}\frac{d\sigma_j}{dt} = q_{yj}(Dv)_j. \end{cases}$$

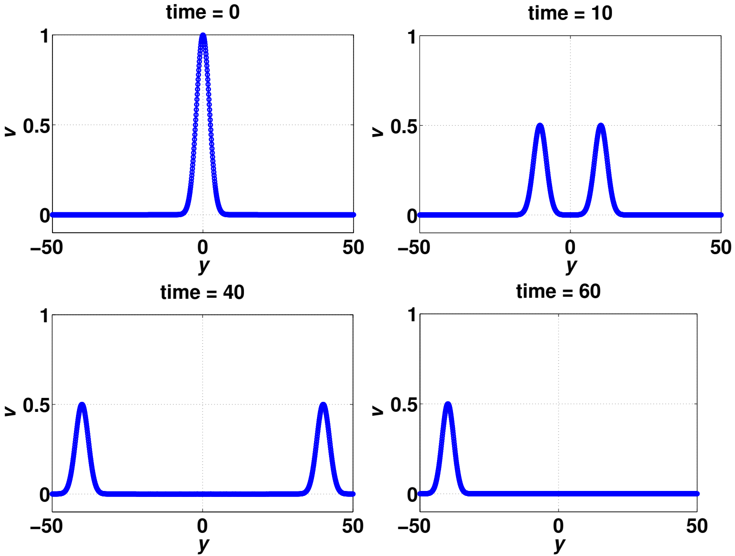
Second order accuracy: $(Dv)_1 = \dfrac{v_2 - v_1}{h}, \quad (Dv)_j = \dfrac{v_{j+1} - v_{j-1}}{2h}.$

Penalty parameters α, β are chosen by requiring stability.

$\alpha = 1, \beta = 0.$

There are also other possible choices !

A simple numerical test



How about fault + friction laws?

Generalize to 3D linear elasticity in heterogeneous media + complex geometries + interface conditions + friction laws

Ongoing and future work

- Fully parallel with MPI
- Automated multi–block
- Rate and state friction law
- Perform more SCEC benchmarks and more tests
- Subduction zone simulations

Suggestions ?