Convergence Tests

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Objectives:

1. Assess accuracy of numerical solutions to earthquake rupture problems
2. For which problems does there exist an *exact* solution?
3. Identify main sources of error (steep gradients, discontinuities, or even singularities in solution)
4. Suggestions for formulating problems with smoother solutions

Accuracy of Numerical Solutions

Many numerical methods approximate exact solution at discrete points and times:

$$q_{ij}^{n}(\Delta x, \Delta y, \Delta t) = q_{exact}\left(x_i, y_j, t_n\right) + O\left(\Delta x^p, \Delta y^p, \Delta t^r\right)$$

Exponents p and r are properties of numerical method *for smooth solutions*. For solutions with discontinuities in fields or their derivatives, exponents are often reduced and depend on nature of solution being approximated.

Define scalar measure of error, at time t_n, using some norm:

$$e_2(\Delta x, \Delta y, \Delta t) = \sqrt{\Delta x \Delta y \sum_{ij} \left| q_{ij}^{n}(\Delta x, \Delta y, \Delta t) - q_{exact}\left(x_i, y_j, t_n\right) \right|^2}$$

$$e_1(\Delta x, \Delta y, \Delta t) = \Delta x \Delta y \sum_{ij} \left| q_{ij}^{n}(\Delta x, \Delta y, \Delta t) - q_{exact}\left(x_i, y_j, t_n\right) \right|$$

$$e_\infty(\Delta x, \Delta y, \Delta t) = \max_{ij} \left| q_{ij}^{n}(\Delta x, \Delta y, \Delta t) - q_{exact}\left(x_i, y_j, t_n\right) \right|$$

Convergence Rates

Assume that $h = \Delta x = \Delta y \propto \Delta t$

Given solution at two resolutions, estimate convergence rate as

$$\frac{\log\left[e(h_1)/e(h_2)\right]}{\log(h_1/h_2)} = \frac{\log\left[\left(ah_1^p + b\Delta t_1^r + \ldots\right)\Big/\left(ah_2^p + b\Delta t_2^r + \ldots\right)\right]}{\log(h_1/h_2)}$$

$$= \frac{\log\left[\left(ah_1^p + \ldots\right)\Big/\left(ah_2^p + \ldots\right)\right]}{\log(h_1/h_2)}$$

(valid for p=r or if temporal errors smaller than spatial errors)

$$= \frac{\log\left[(h_1/h_2)^p + \ldots\right]}{\log(h_1/h_2)}$$

$$= p + \ldots$$

For sufficiently refined grids, this estimate will yield smaller of p and r.

What if exact solution is not known?

Define $$E(h_1,h_2)=\left\|q(h_1)-q(h_2)\right\|$$
$$=\left\|\left(q_{exact}+\alpha h_1^p+\dots\right)-\left(q_{exact}+\alpha h_2^p+\dots\right)\right\|$$
$$=\left\|\alpha\left(h_1^p-h_2^p\right)+\dots\right\|$$
$$=a\left(h_1^p-h_2^p\right)+\dots$$

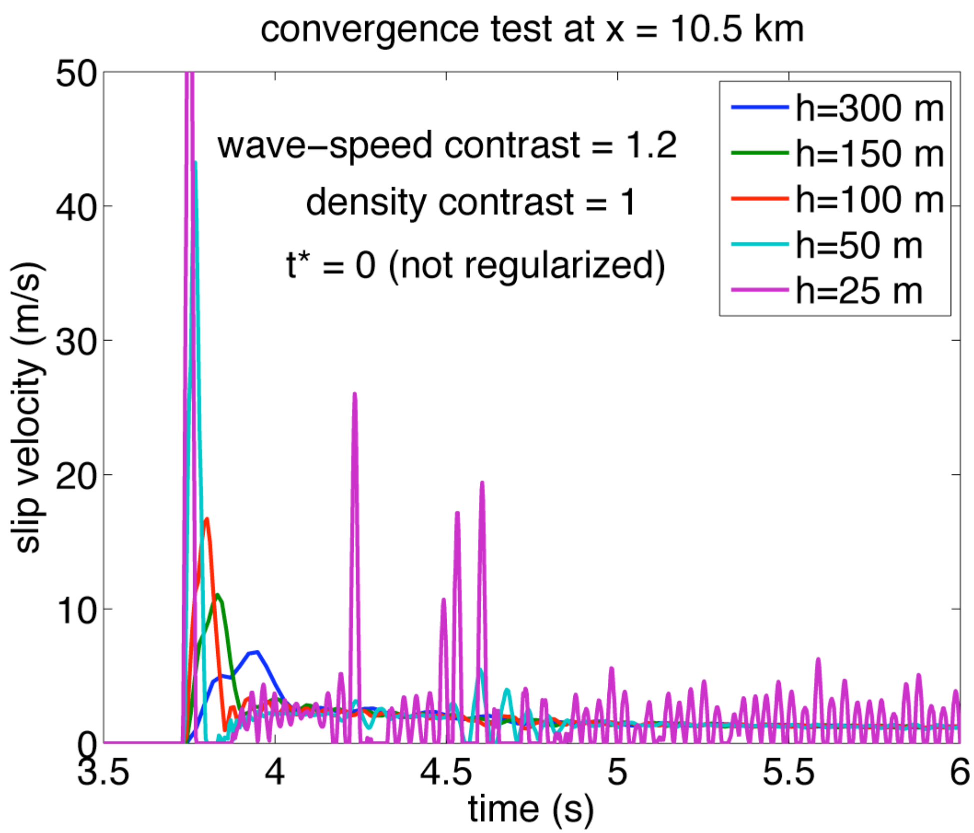
Given solution at three resolutions, with $h_2/h_1=h_3/h_2=R$, estimate convergence rate as

$$\frac{\log\left[E(h_1,h_2)/E(h_2,h_3)\right]}{\log(h_1/h_2)}=\frac{\log\left[\dfrac{a\left(h_1^p-h_2^p\right)+\dots}{a\left(h_2^p-h_3^p\right)+\dots}\right]}{\log(h_1/h_2)}=\frac{\log\left[\dfrac{h_1^p\left(1-R^p\right)+\dots}{h_2^p\left(1-R^p\right)+\dots}\right]}{\log(h_1/h_2)}=p+\dots$$

Well-posed and Ill-posed Problems

To have convergence, problem must have a solution (i.e., be well-posed). Several SCEC validation problems are ill-posed:

TPV7: low-contrast bimaterial problem with slip-weakening friction (sliding at constant *f* unstable, growth rate of Fourier mode proportional to wavenumber)



Well-posed and Ill-posed Problems

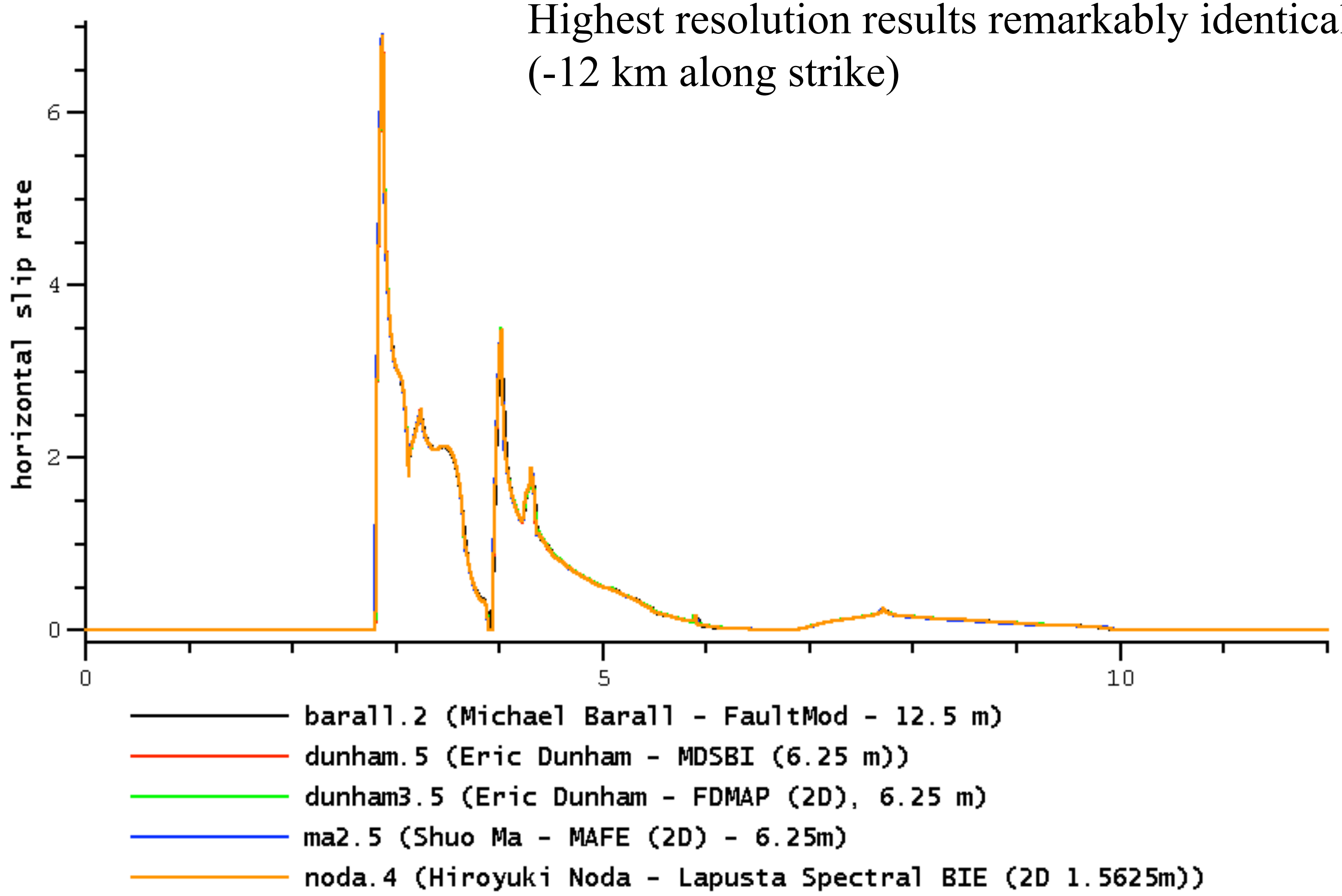
To have convergence, problem must have a solution (i.e., be well-posed).
Several SCEC validation problems are ill-posed:

TPV13: rate-independent plasticity (under certain conditions, wave speeds during plastic deformation decrease to zero, then become complex $\Rightarrow$ equations become elliptic in time, shear localization)

New problems, TPV205/210, are well-posed.
TPV205: standard benchmark in community, simple geometry
TPV210: tough problem, complex interaction of waves with free surface, supershear transition occurs halfway up-dip in 2D version

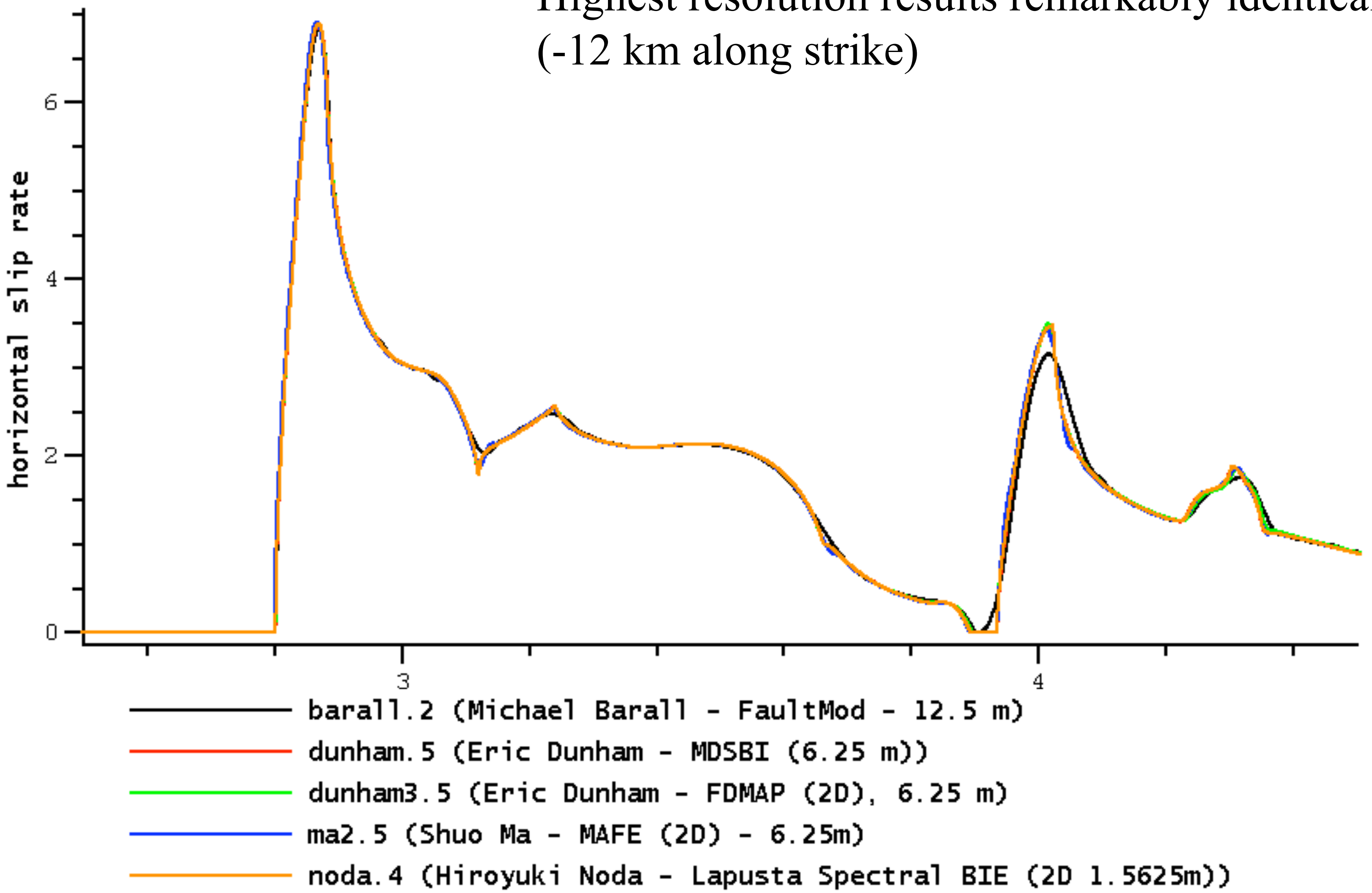
TPV205-2D: Highest resolution for all codes

Highest resolution results remarkably identical!
(-12 km along strike)



TPV205-2D: Highest resolution for all codes

Highest resolution results remarkably identical!
(-12 km along strike)



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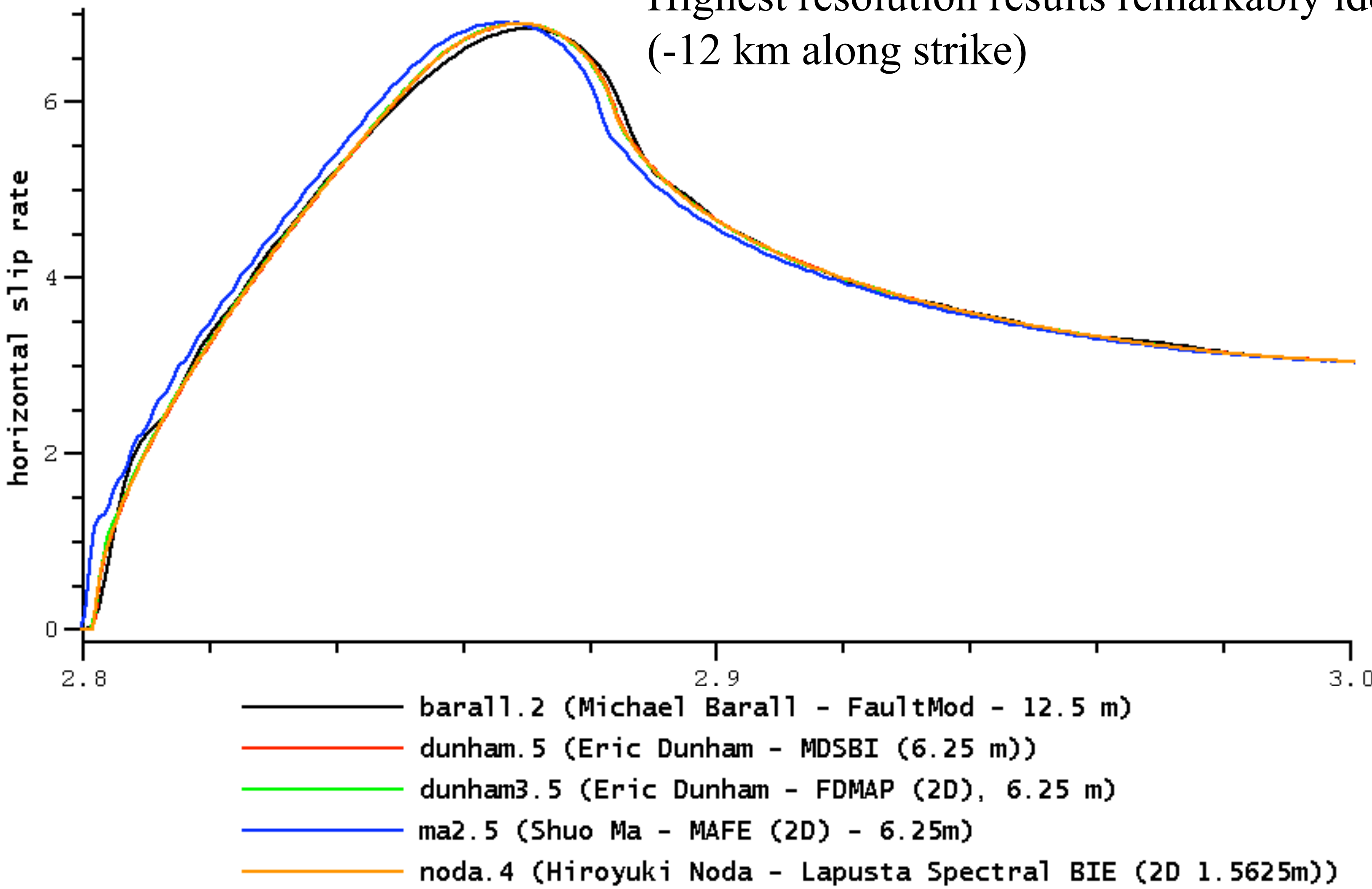
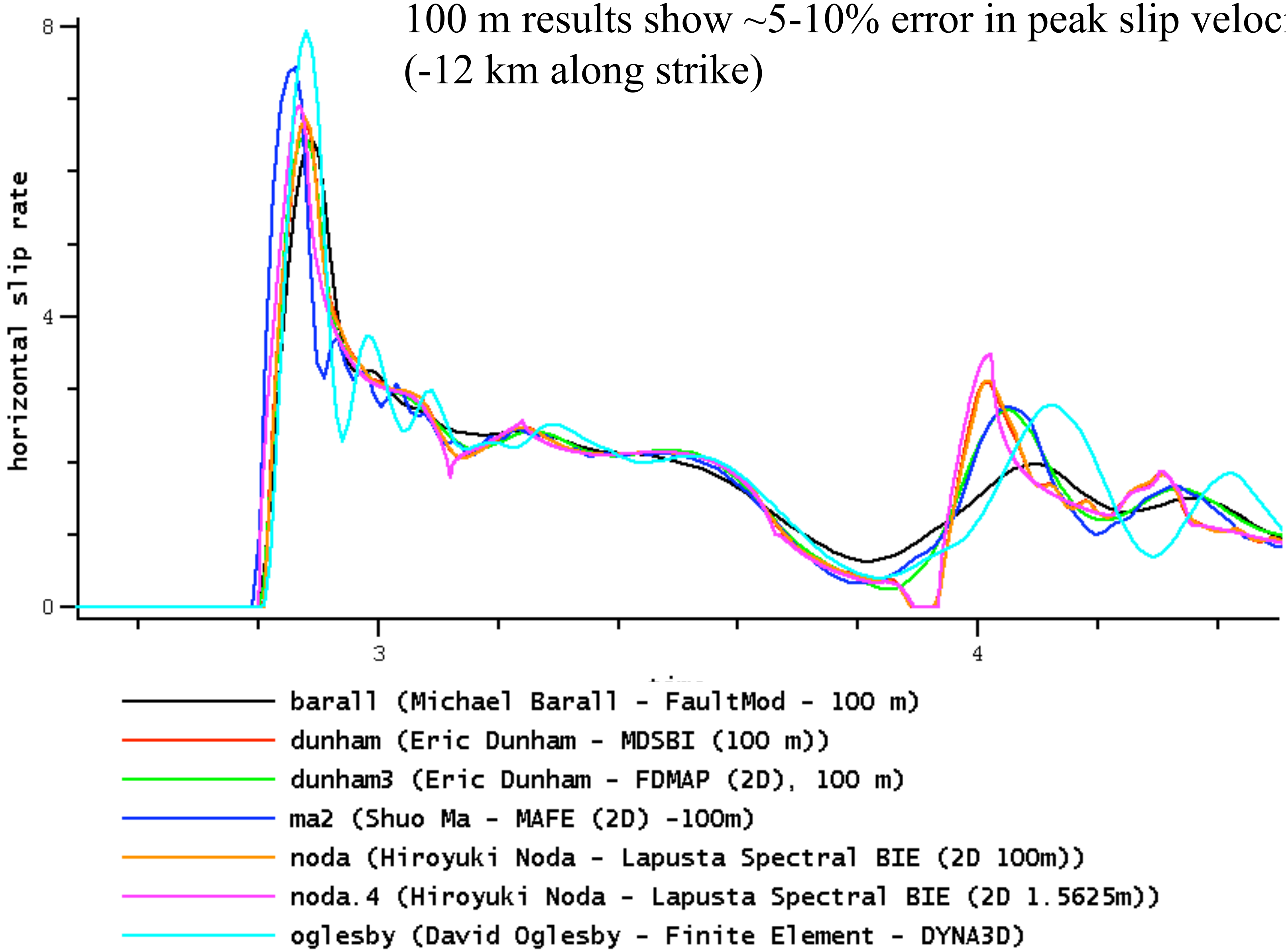


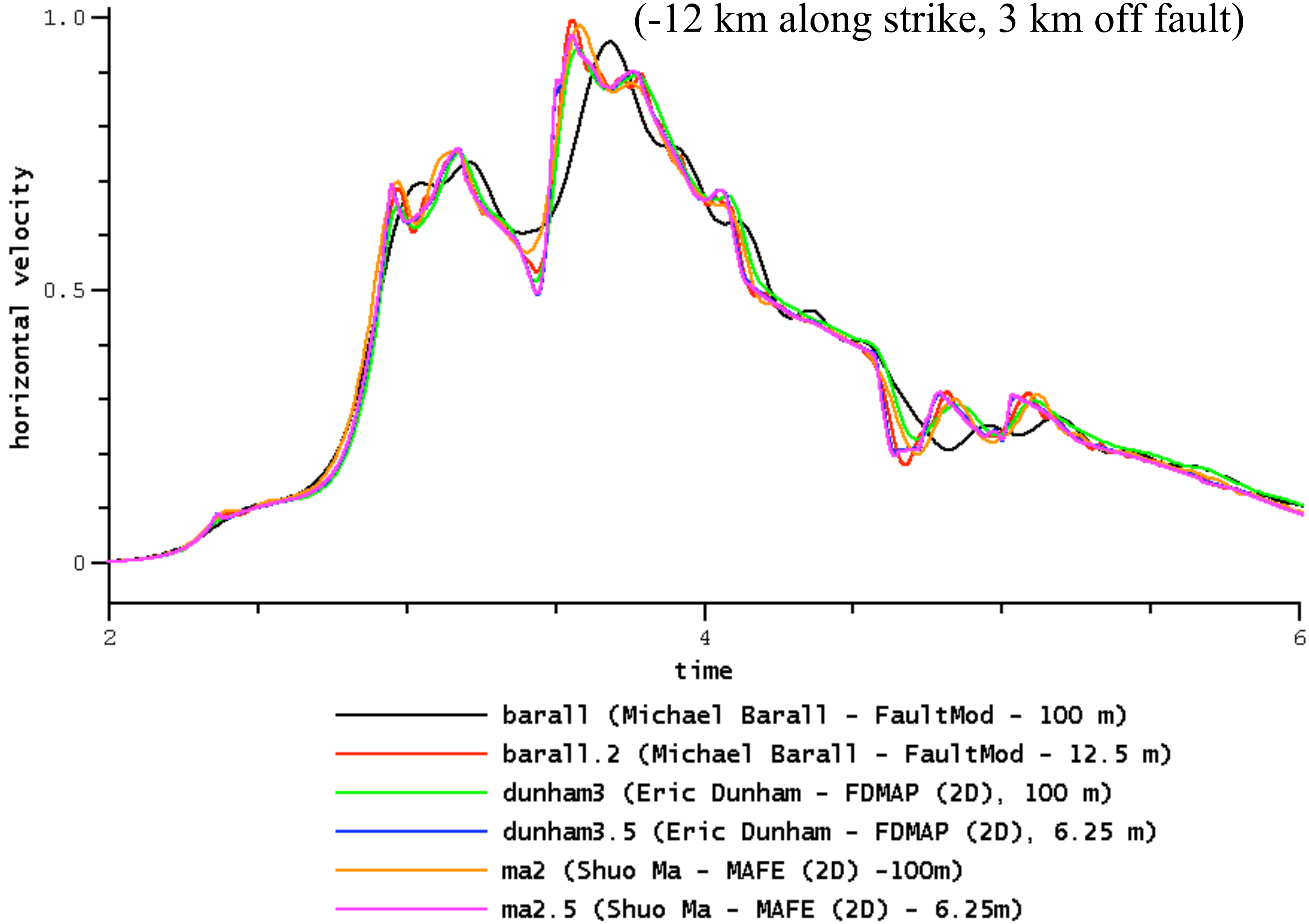
TPV205-2D: 100 m for all codes (+reference)

100 m results show ~5-10% error in peak slip velocity (-12 km along strike)



TPV205-2D: 100 m for all codes (+reference)

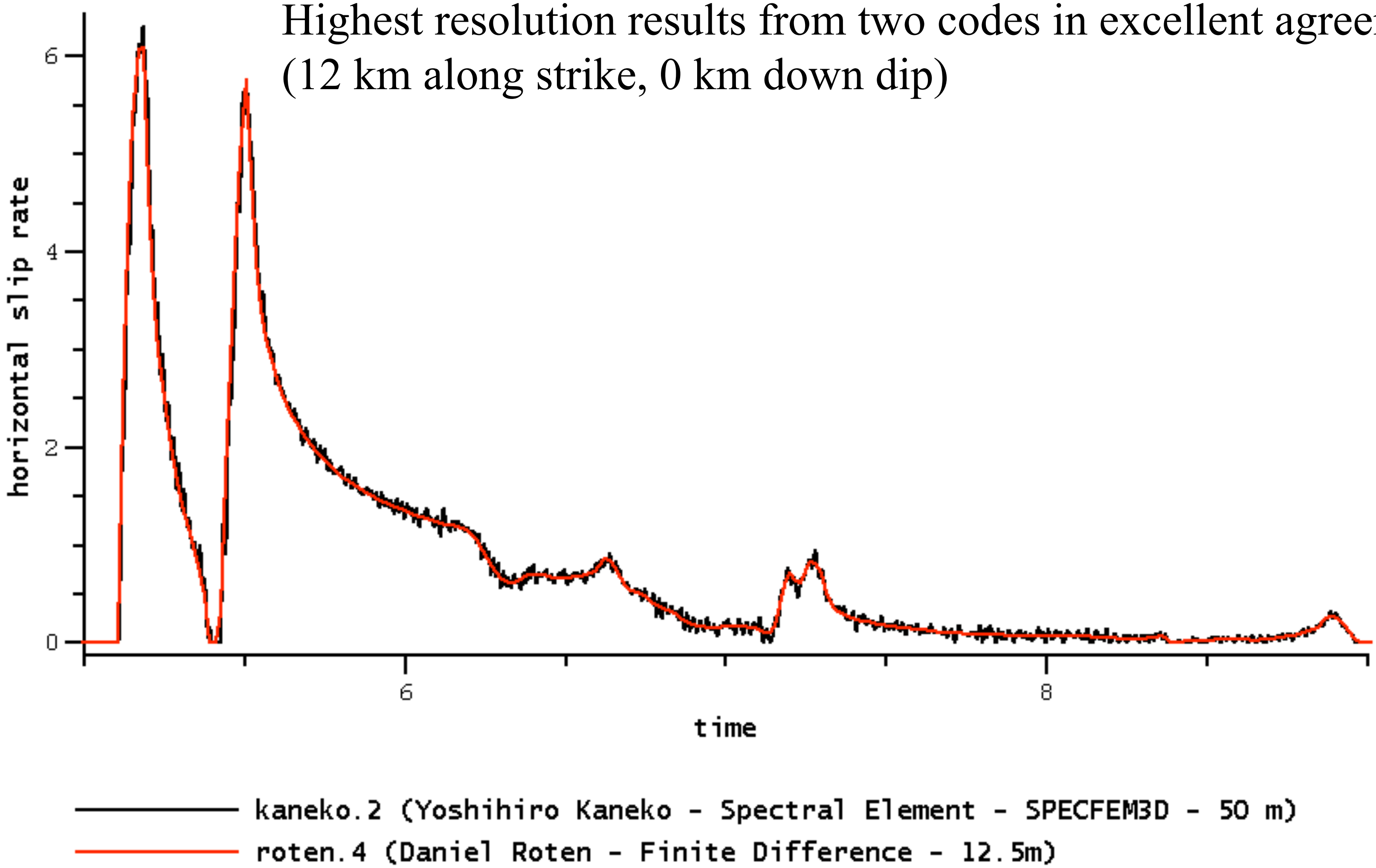
100 m results show ~5-10% error in peak velocity
(-12 km along strike, 3 km off fault)



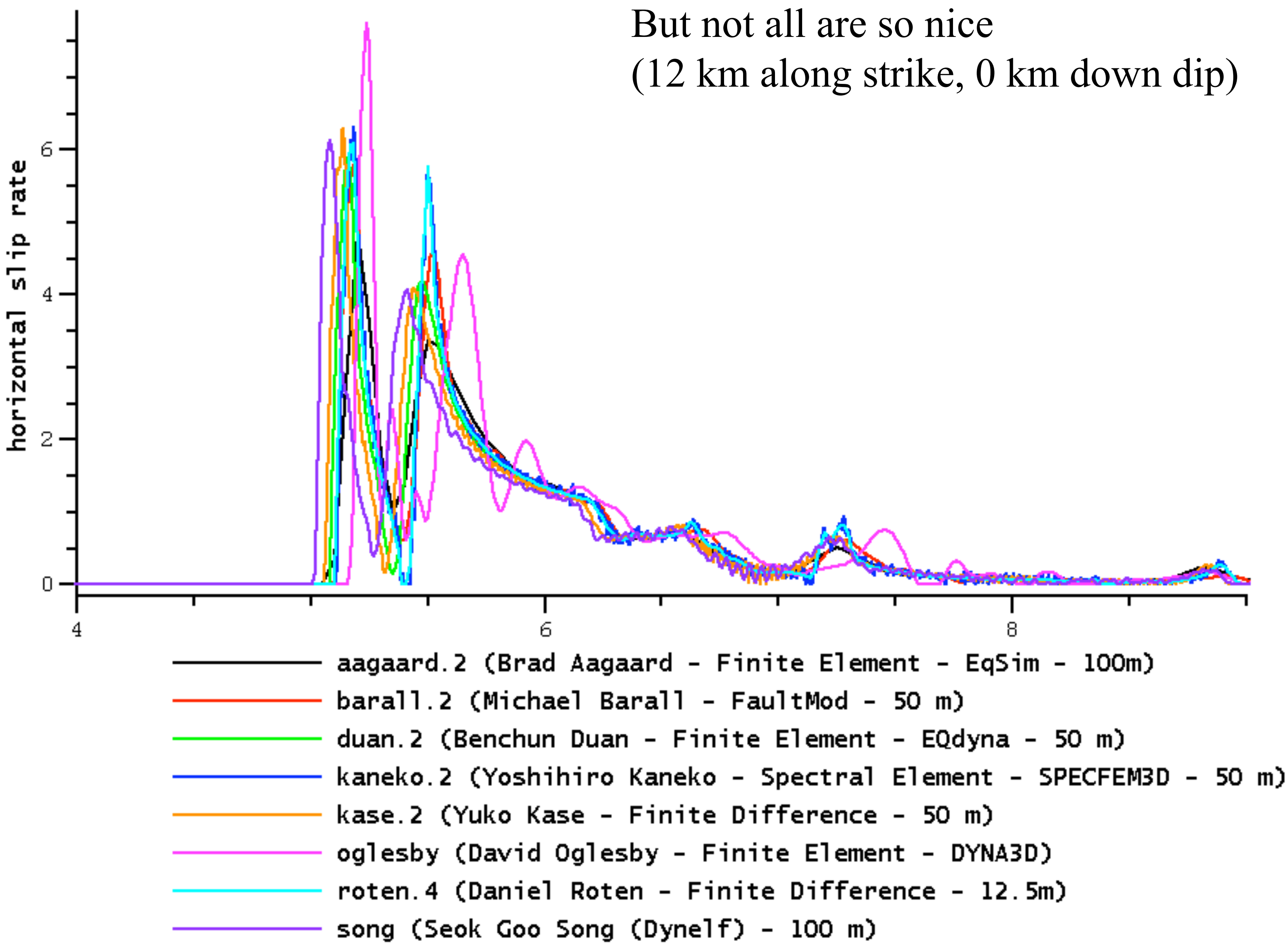
TPV205-3D

3D more challenging, only one group below 50 m grid spacing

Highest resolution results from two codes in excellent agreement
(12 km along strike, 0 km down dip)

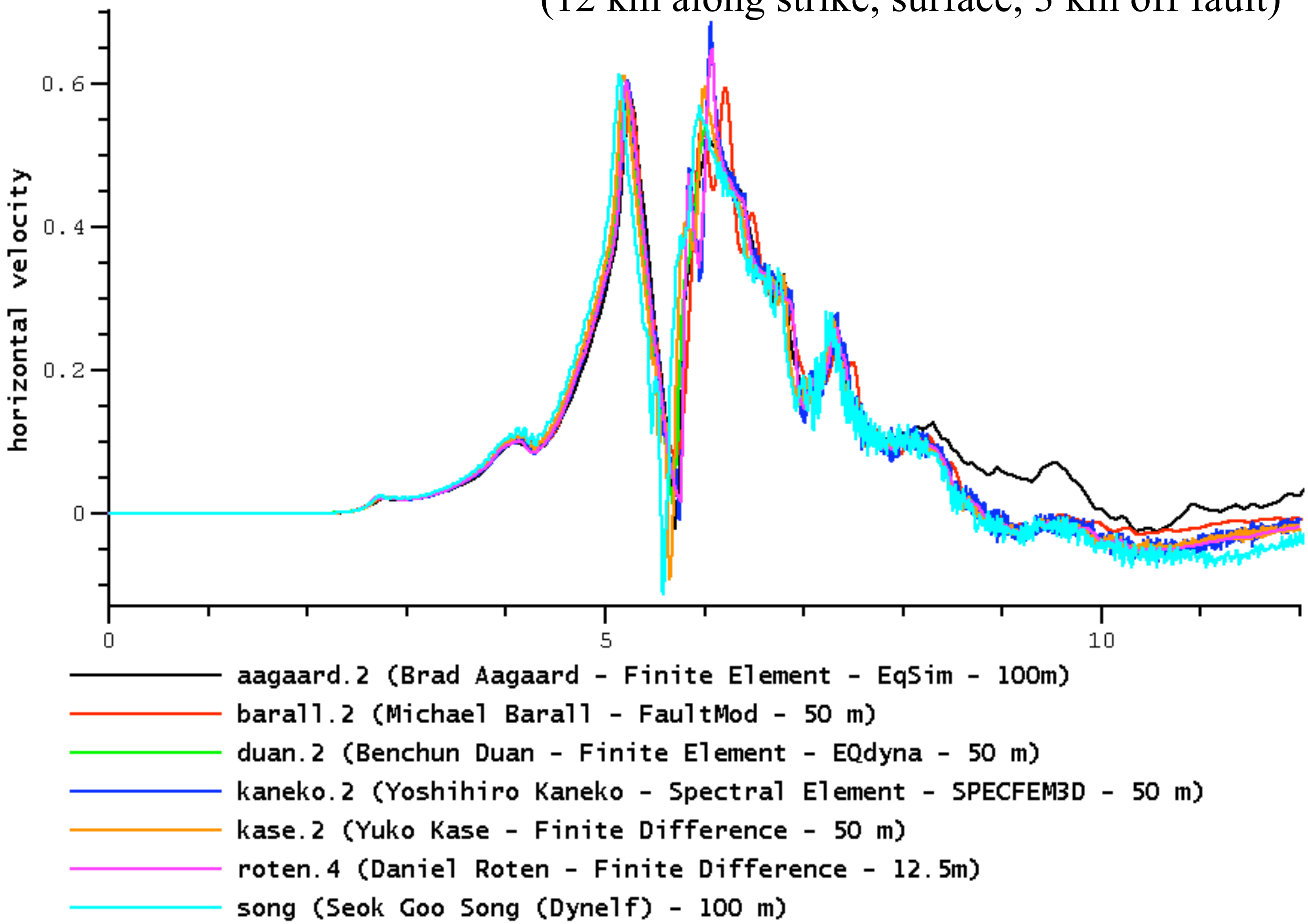


TPV205-3D: Highest resolution for all codes

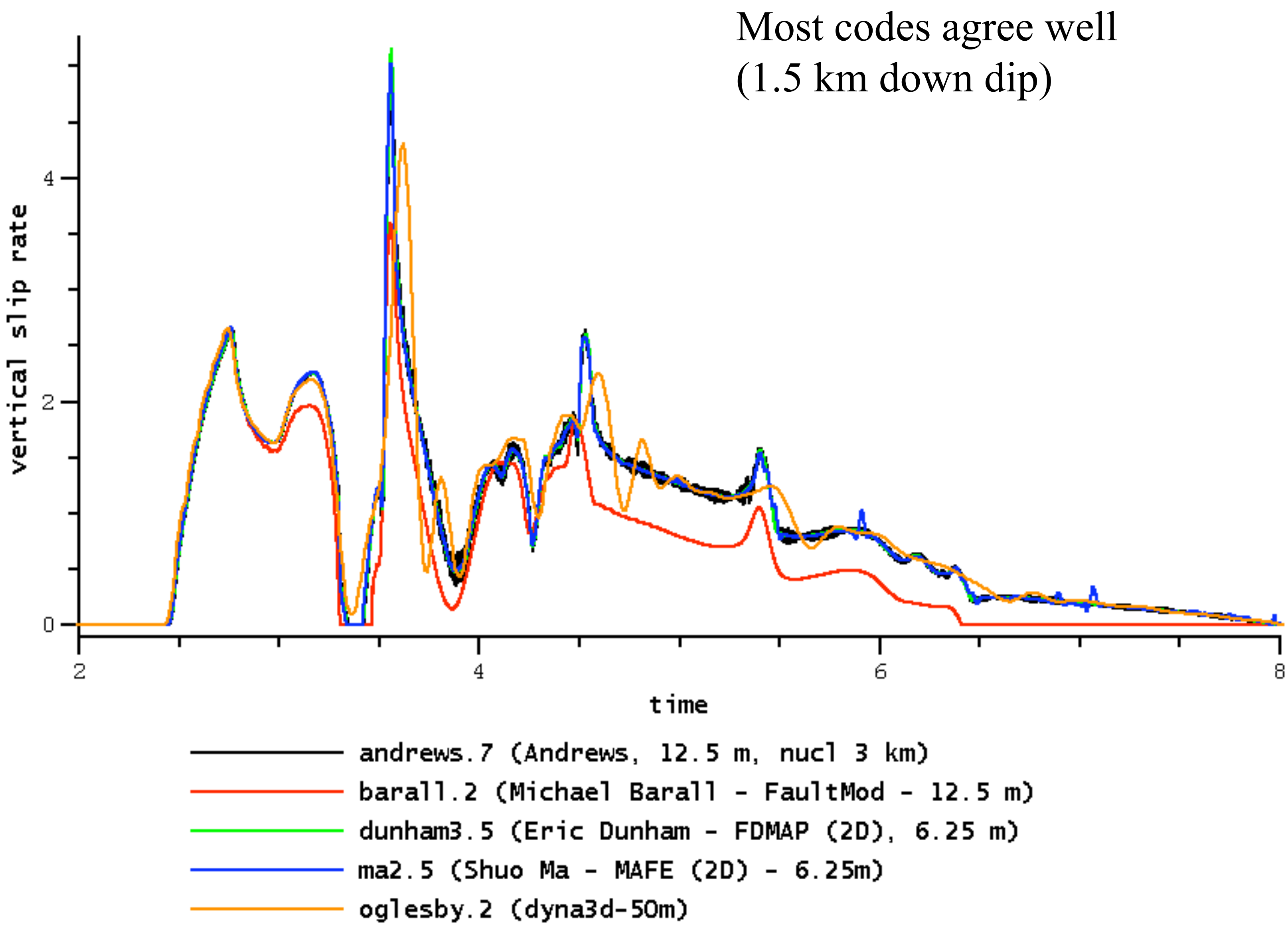


TPV205-3D: Highest resolution for all codes

Ground motion better
(12 km along strike, surface, 3 km off fault)

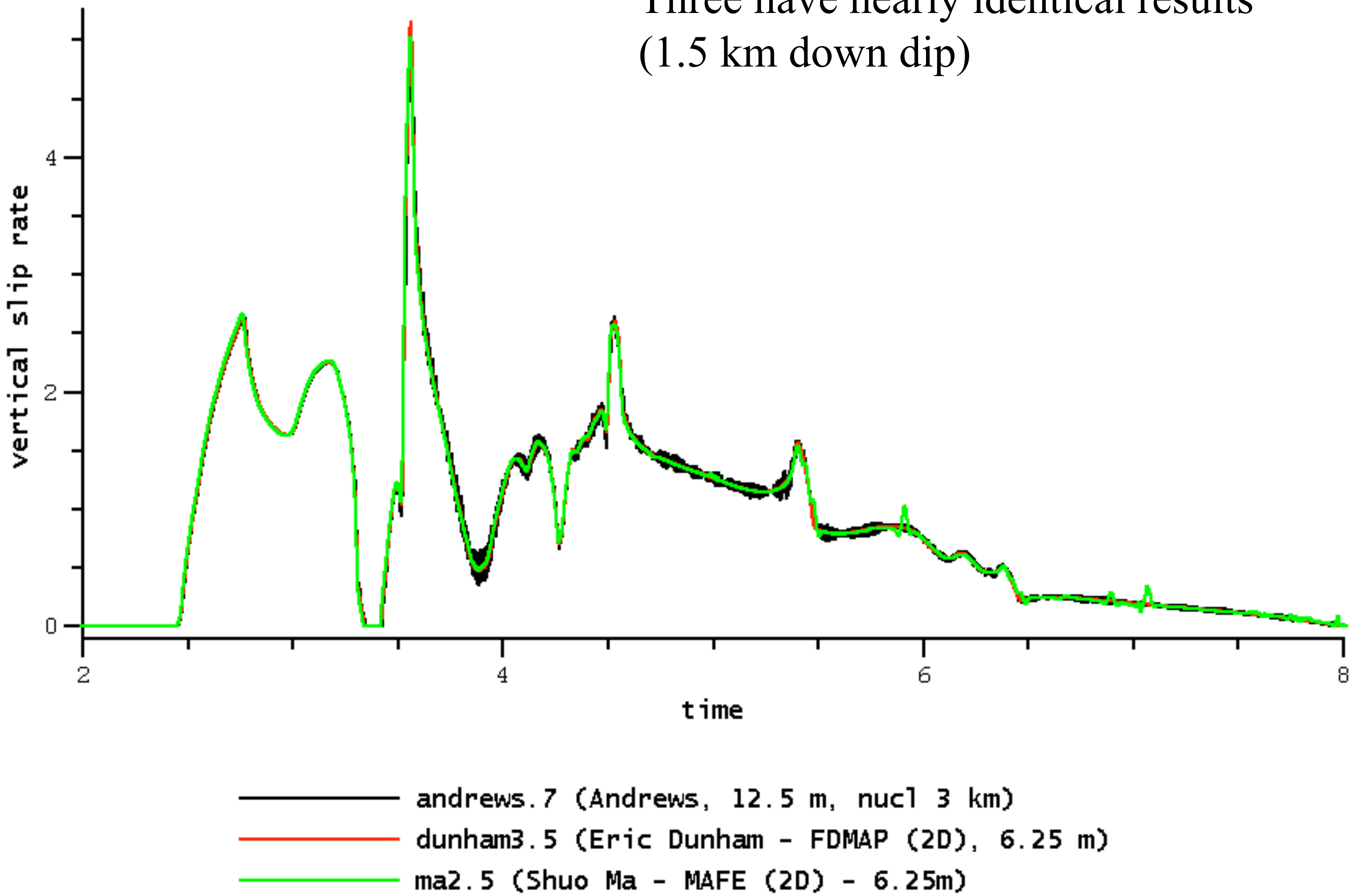


TPV210-2D: Highest resolution for all codes



TPV210-2D

Three have nearly identical results
(1.5 km down dip)



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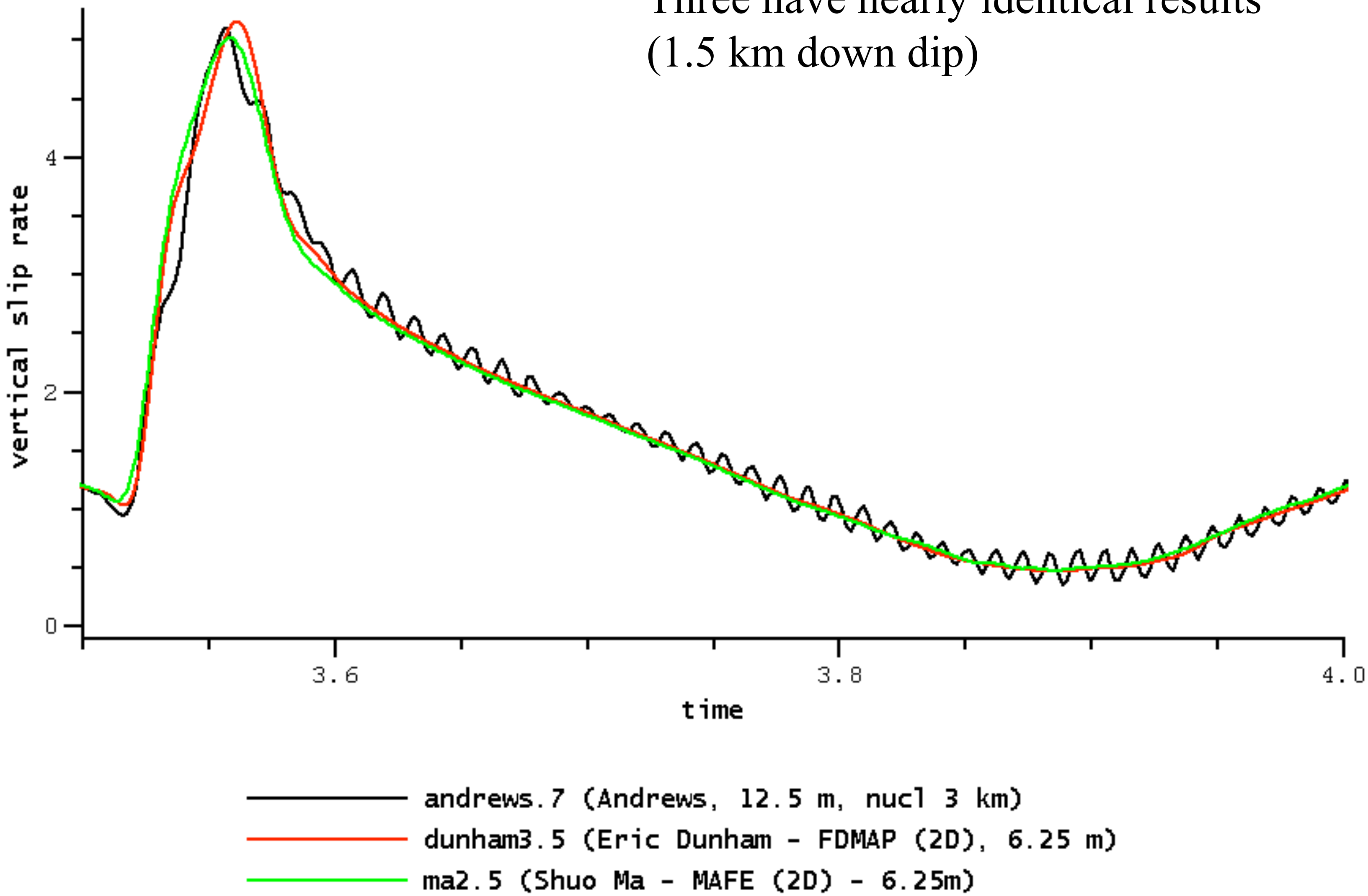
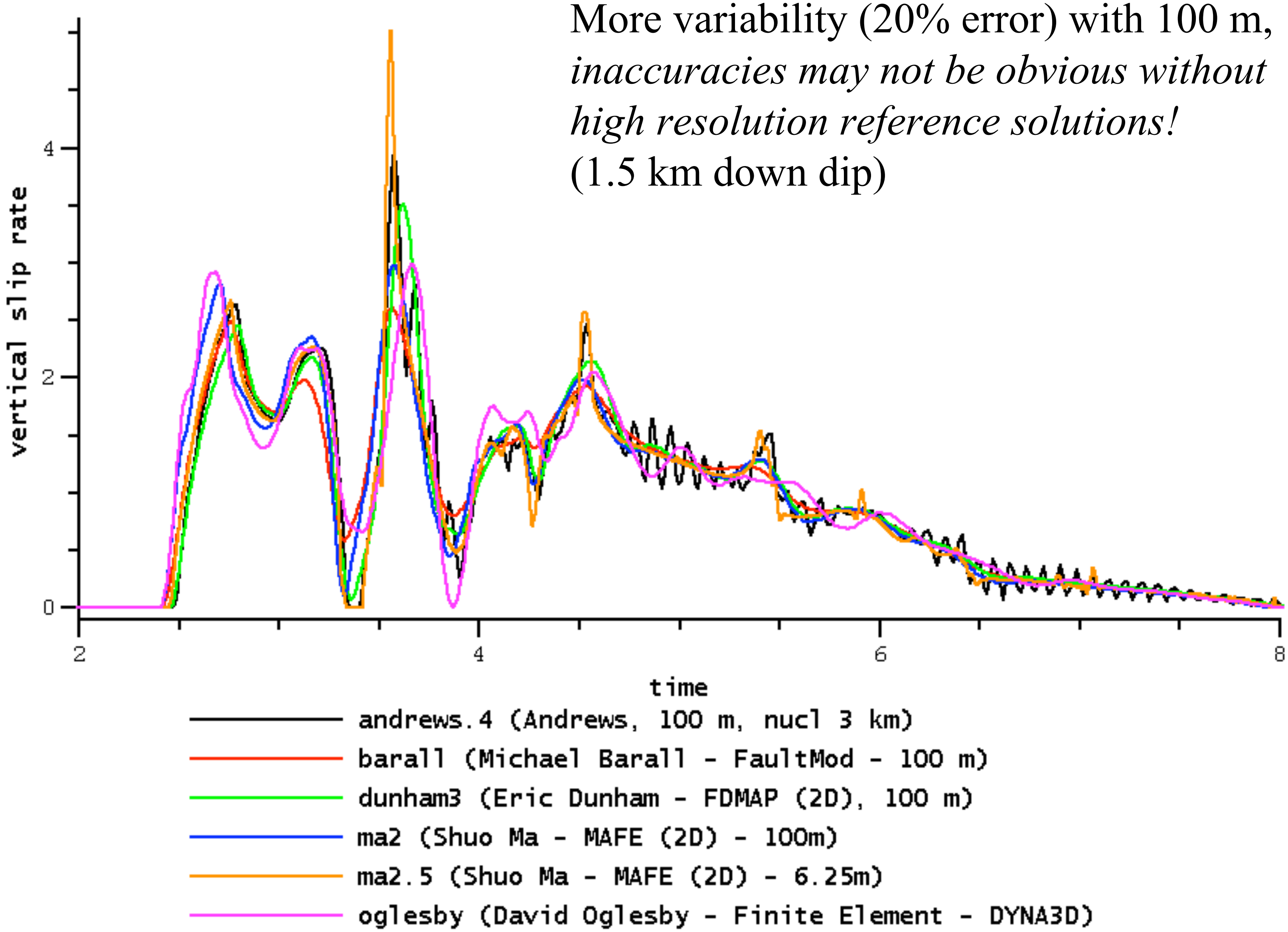
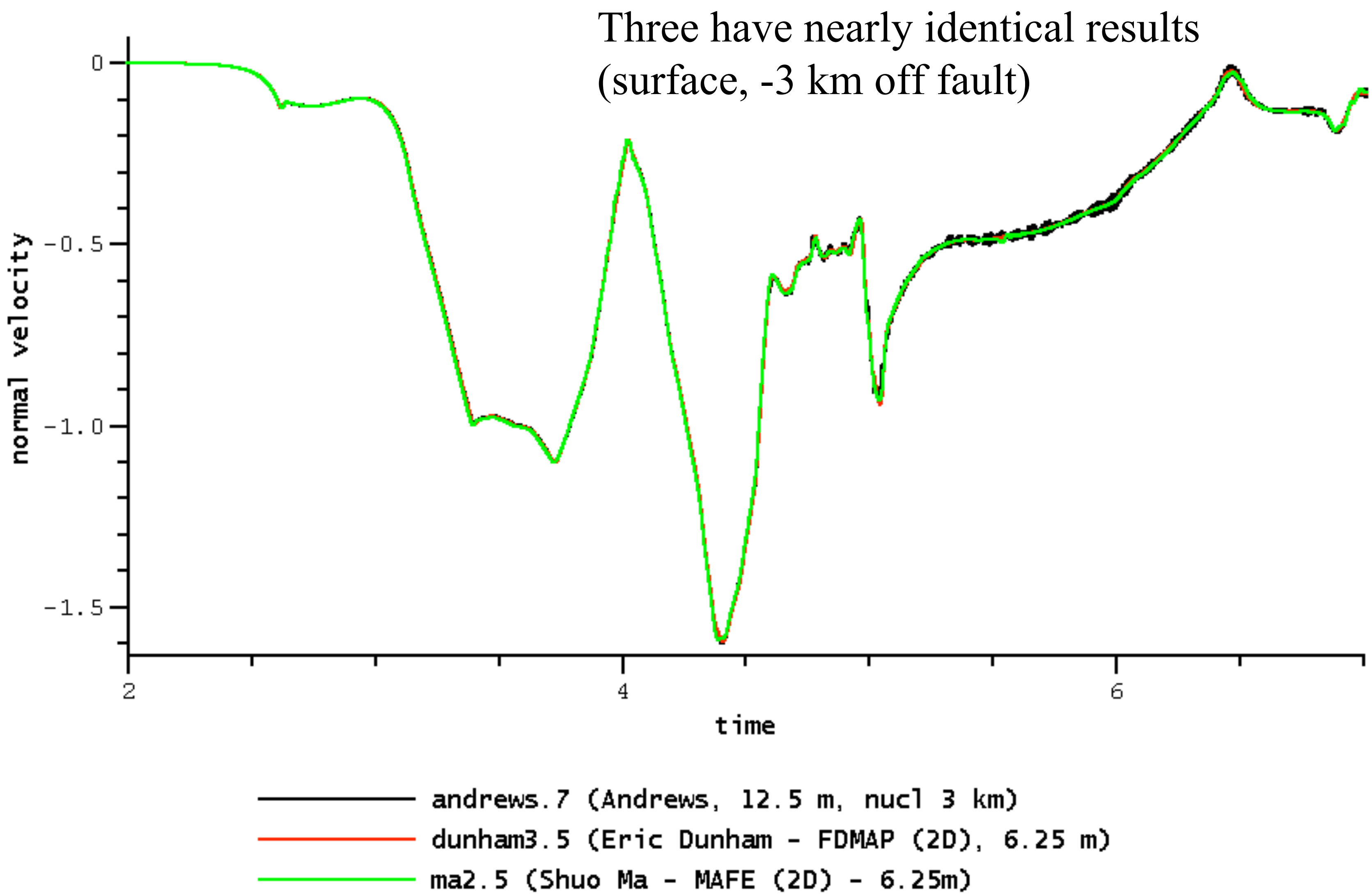


TPV210-2D: 100 m for all codes (+reference)

More variability (20% error) with 100 m, *inaccuracies may not be obvious without high resolution reference solutions!* (1.5 km down dip)

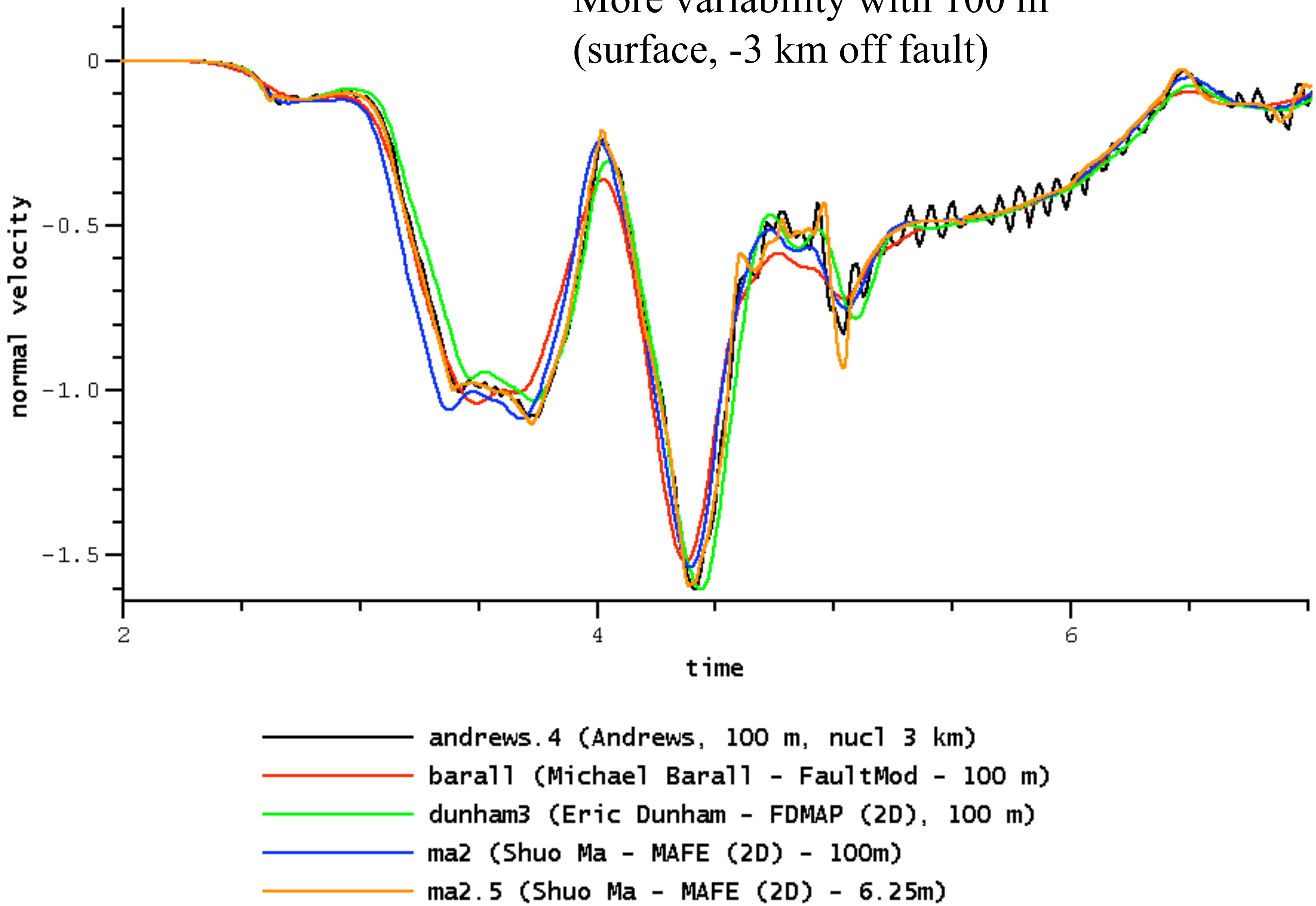


TPV210-2D


Three have nearly identical results
(surface, -3 km off fault)
0
-0.5
-1.0
-1.5
normal velocity
2
4
6
time
andrews.7 (Andrews, 12.5 m, nucl 3 km)
dunham3.5 (Eric Dunham - FDMAP (2D), 6.25 m)
ma2.5 (Shuo Ma - MAFE (2D) - 6.25m)

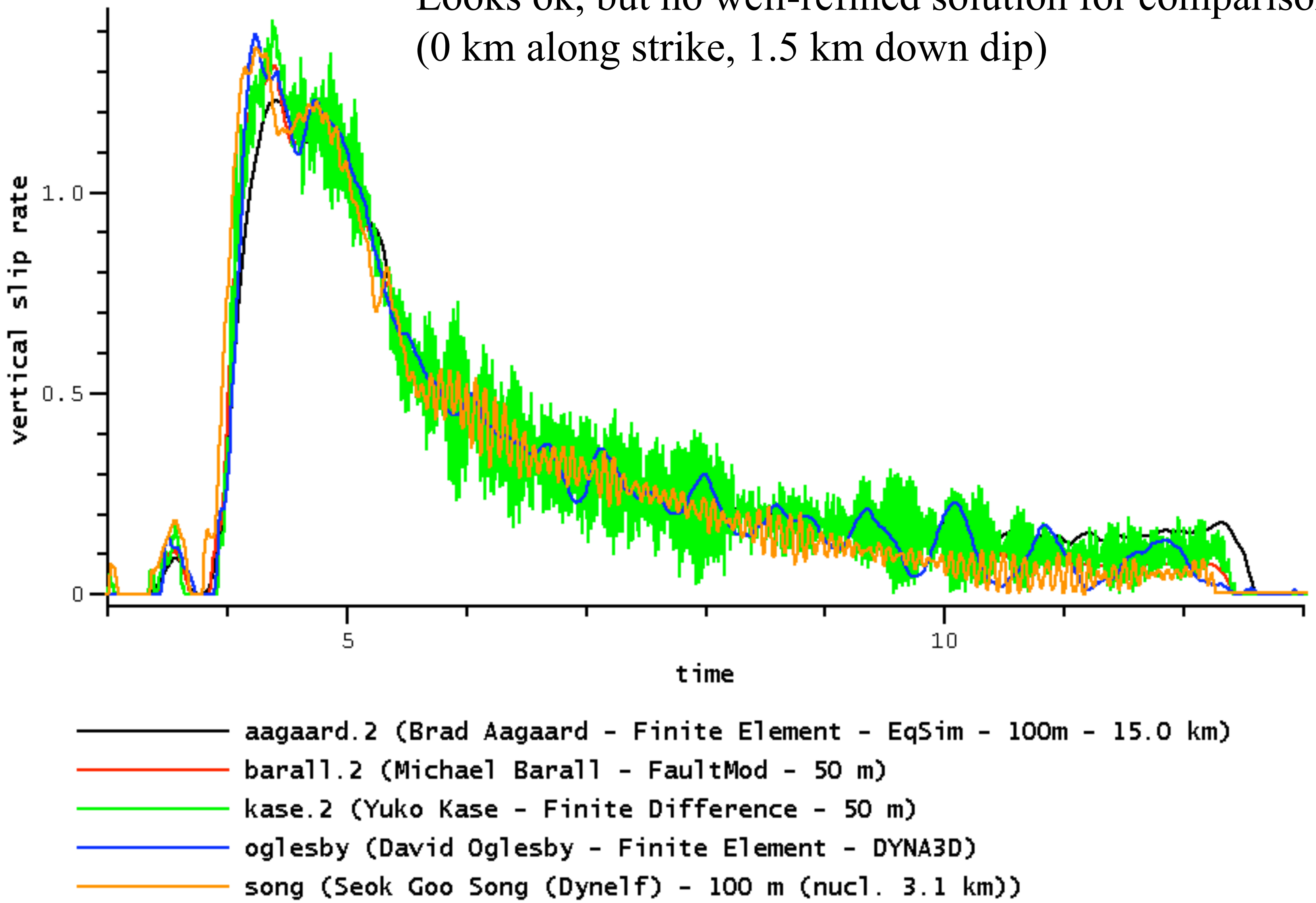
TPV210-2D : 100 m for all codes (+reference)

More variability with 100 m
(surface, -3 km off fault)



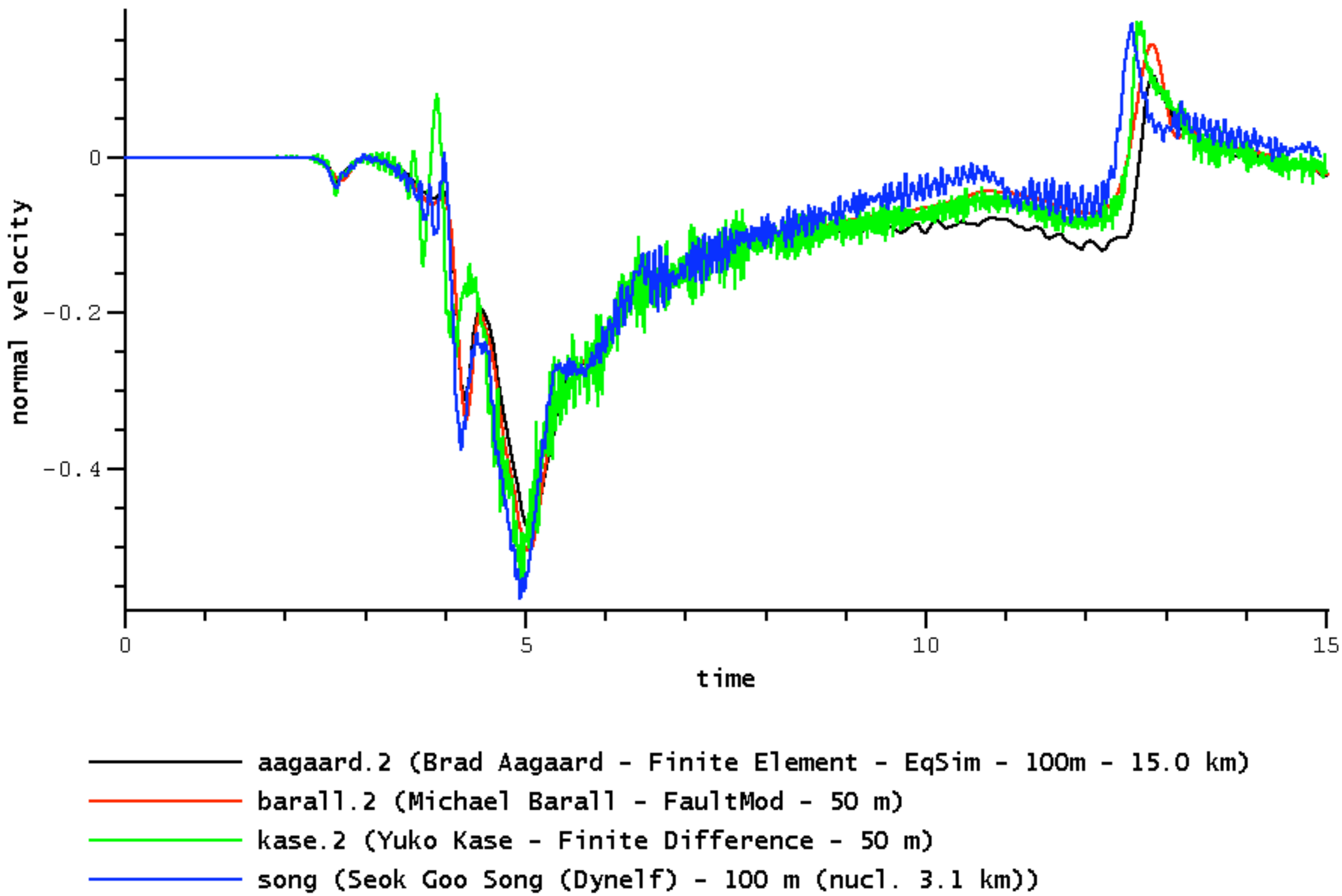
TPV210-3D

Looks ok, but no well-refined solution for comparison
(0 km along strike, 1.5 km down dip)



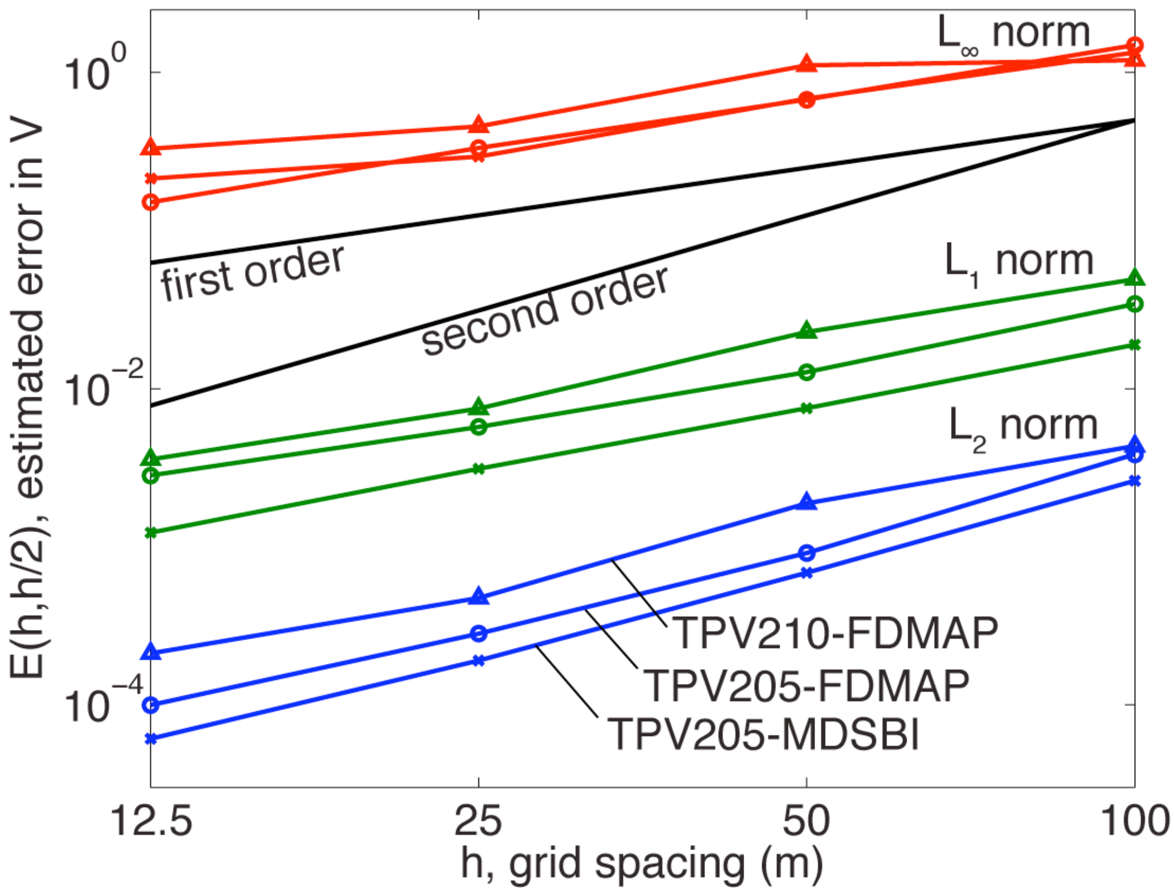
TPV210-3D

(0 km along strike, surface, -3 km off fault)



Error Estimates and Convergence Rates

Error metrics and rate estimates defined previously, for TPV205-2D (MDSBI and FDMAP) and TPV210-2D (FDMAP):



MDSBI (spectral in space, second order in time) and FDMAP (fourth order in space and time) exhibit only first order convergence in L_1 and L_2 norms, and worse in L_∞ norm!

Error Estimates and Convergence Rates

Theoretical convergence rates are *not* expected for TPV205 or TPV210:

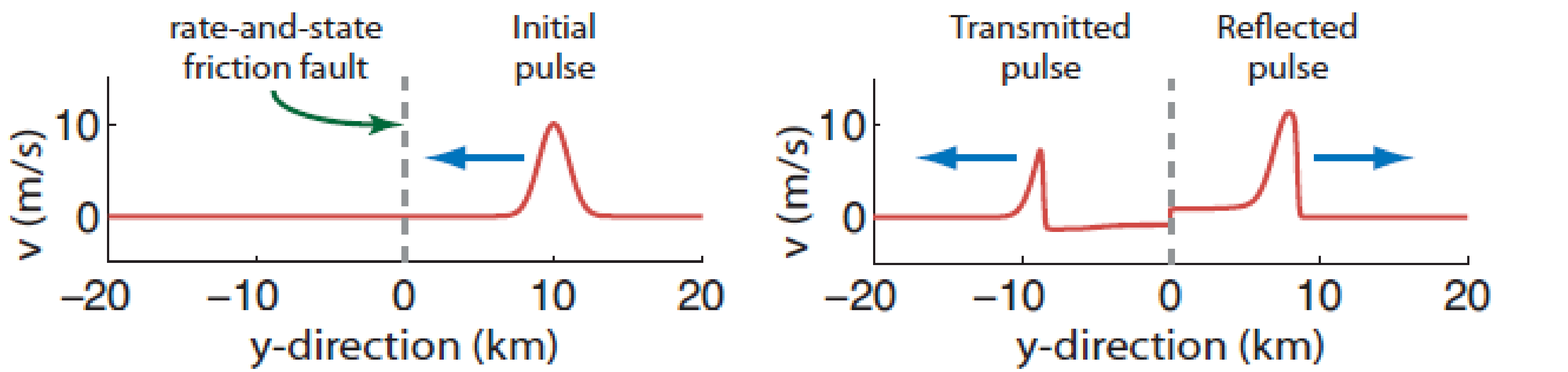
• slip-weakening friction introduces discontinuities in first derivatives of fields

• boxcar-shaped nucleation causes O(Δx) errors

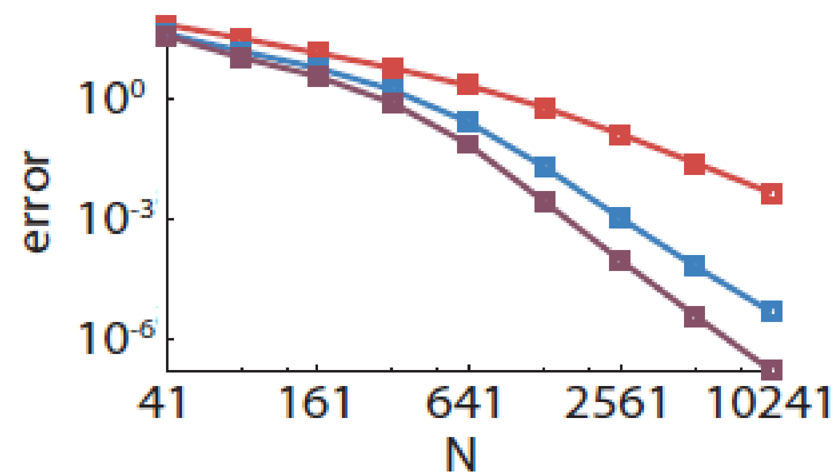
• abrupt termination of faults with infinite strength barriers can cause $r^{-1/2}$ stress singularities (even with slip-weakening friction)

Error Estimates and Convergence Rates

Perhaps better (for testing convergence, at least) to use rate-and-state friction with smooth initial conditions and termination of ruptures? True in 1D:

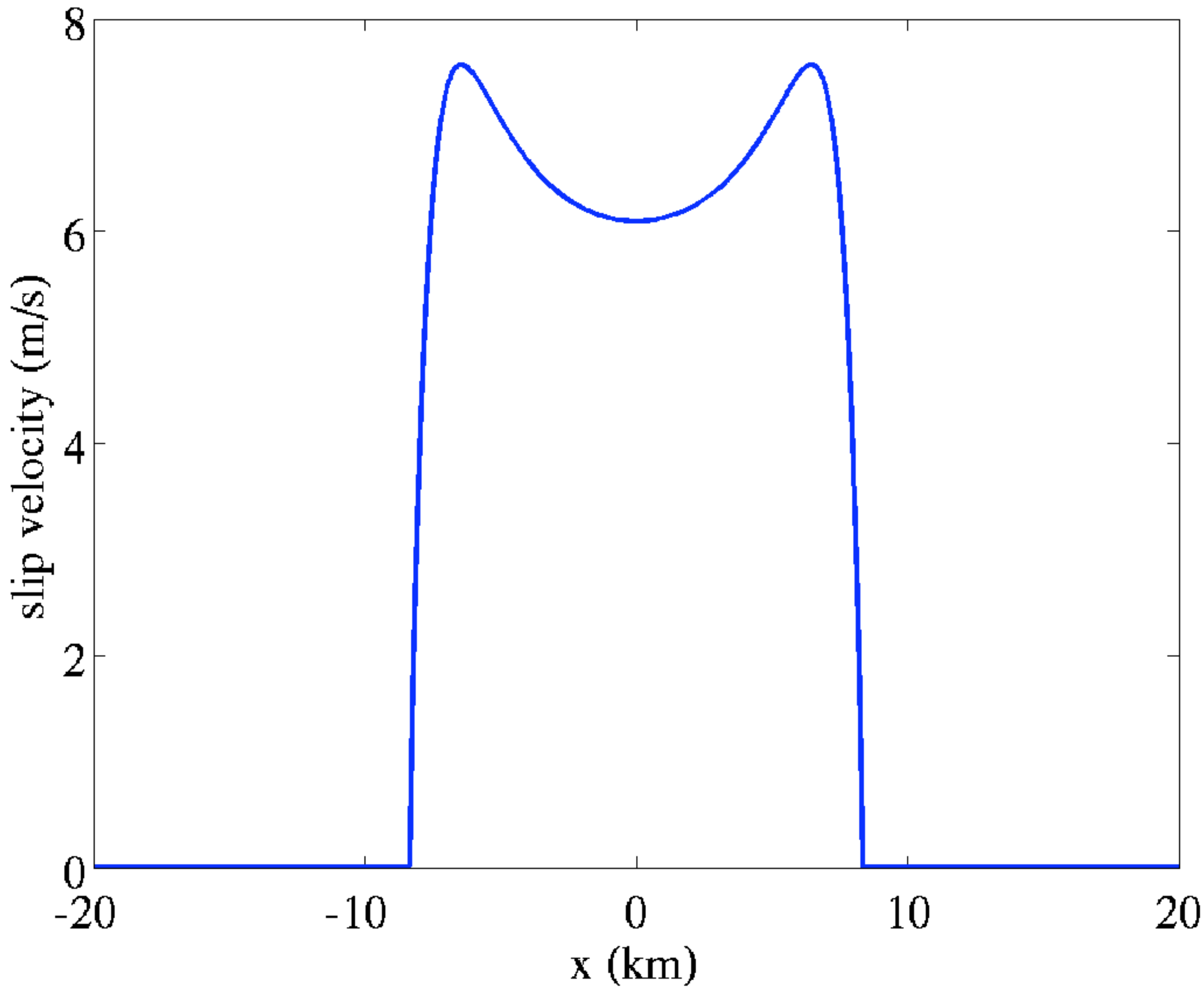


N =	2nd Order	3rd Order	4th Order
41	1.12	1.40	1.79
81	1.19	1.38	1.53
161	1.25	1.82	2.23
321	1.46	2.69	3.46
641	1.82	3.74	4.76
1281	2.22	4.24	4.89
2561	2.45	4.08	4.67
5121	2.50	3.81	4.58



[study by Jeremy Kozdon, Dunham, and Nördstrom, 2009]

Even for smooth solutions, low-order convergence often realized at typical resolutions because solutions appear to have discontinuous first derivatives.



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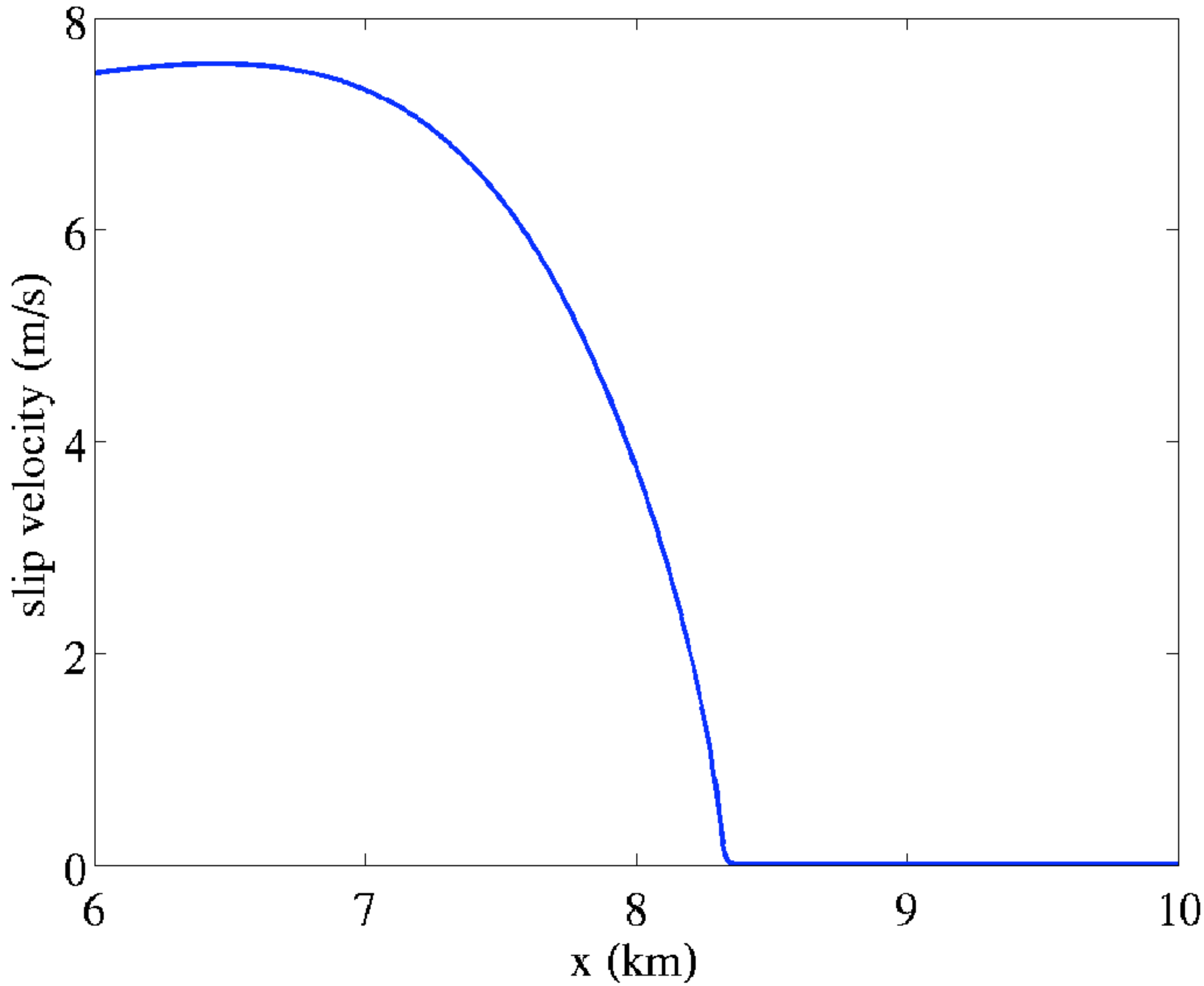

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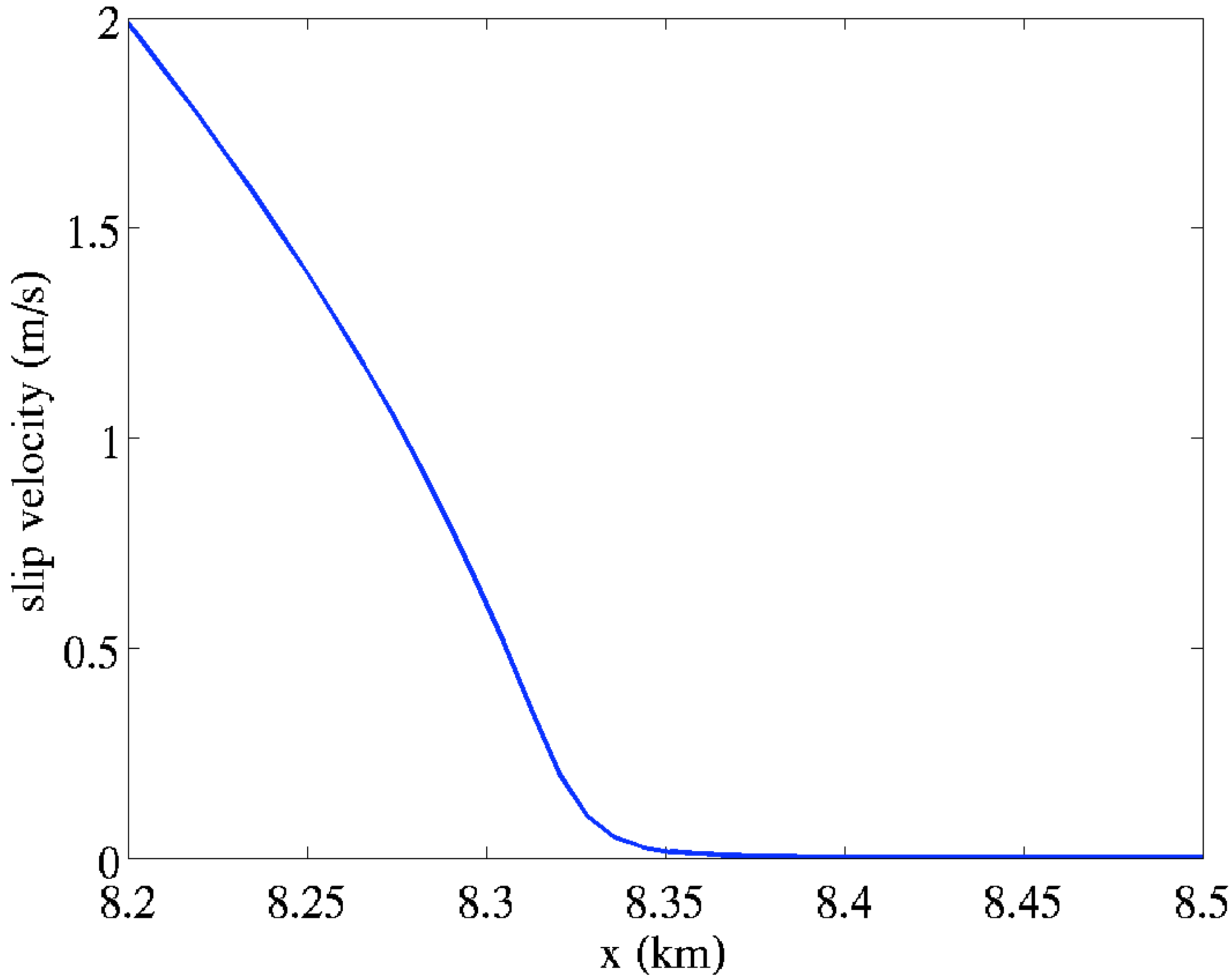

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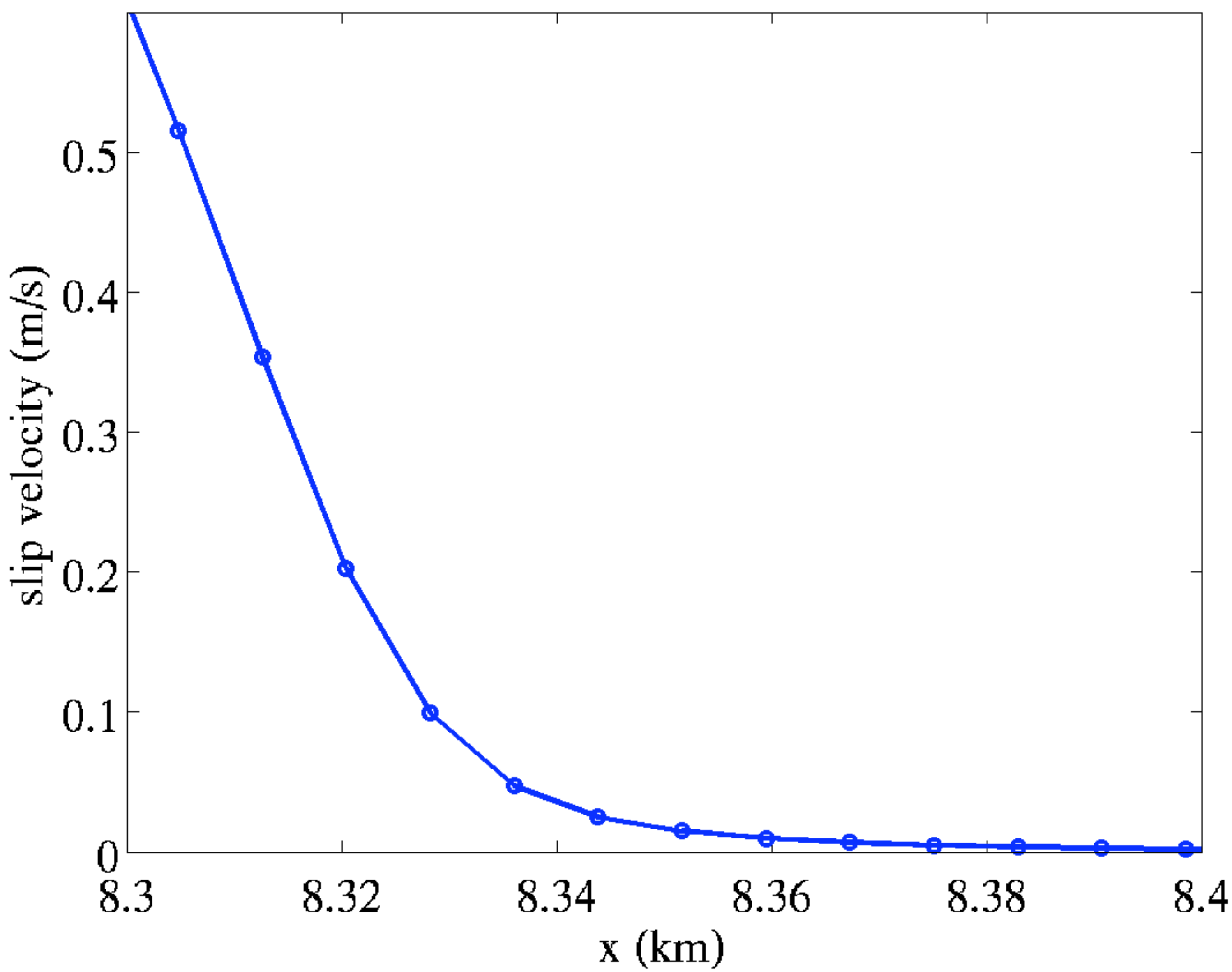

Suggestions

Code validation website could offer error and convergence rate estimation (no interpolation required unless one solution is deemed “exact” or refinement ratio not constant)

Terminate ruptures more gradually (without stress singularity) by either

- continuously lowering initial stress beyond fault edges
- continuously increasing strength (or having constant finite strength) beyond fault edges

Ambiguities with representing discontinuous functions (and O(Δx) errors) -- see next slide -- eliminated by using smooth functions instead (e.g., Gaussian)

Use regularized plasticity formulation?

TPV210-2D: Convergence Issues

Discretization of discontinuous function influences accuracy (1.5 km down dip)