Time Stepping for Earthquake Cycles with Plasticity

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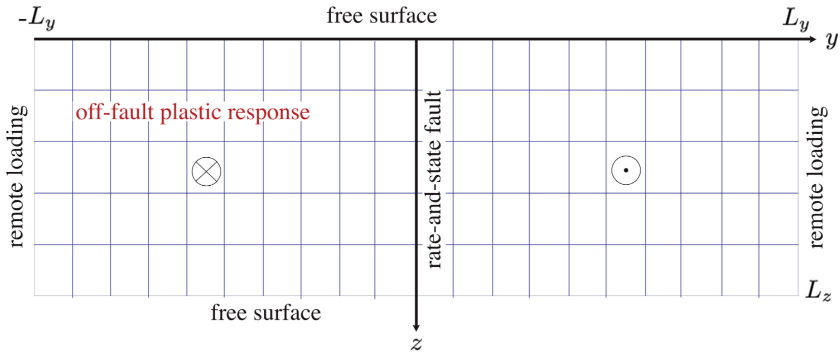
Motivation for this study

- To be able to incorporate plastic response throughout the entire earthquake cycle.
- Explore how plastic strain influences subsequent events.
- For this talk: to share insight into time-stepping methodology for plasticity in SEAS models.

Details in Erickson et al. (**JMPS**, 2017).

Schematic for antiplane shear problem



Problem set up (Equilibrium Equation for Antiplane Strain):

Equilibrium:

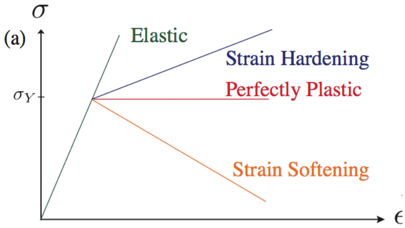
$$0 = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z}, \qquad (x, z) \in [0, L_x] \times [0, L_z] \tag{1}$$

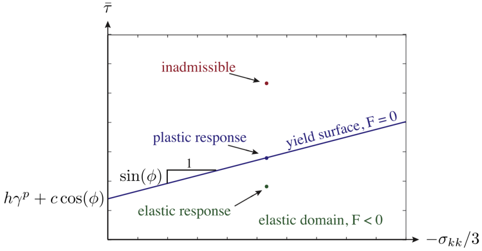
Boundary Conditions:

$$\begin{aligned} u(0, z, t) &= \delta(z, t)/2 \quad \text{(enforce slip)} \\ u(L_x, z, t) &= V_p t/2 \quad \text{(remote displacement)} \\ \sigma_{yz}(x, 0, t) &= 0 \quad \text{(Earth's free surface)} \\ \sigma_{yz}(x, L_z, t) &= 0 \quad \text{(free surface)} \end{aligned}$$

where u is the out-of-plane displacement. Fault is at boundary $x = 0$.

Stress-Strain Relations





Plastic Constitutive Relations

Hooke's Law:

$$\sigma_{ij} = C_{ijkl}(\epsilon_{kl} - \epsilon^p_{kl})$$

Elastic domain in stress space: $E_\sigma = \left\{\sigma_{ij} : F(\sigma_{ij}) \leq 0\right\}$ where

$F(\sigma_{ij})$ is the yield function.

Flow rule:

$$\dot{\epsilon}^p_{ij} = \lambda P_{ij}(\sigma_{ij})$$

for plastic strain rate tensor P_{ij}.

Drucker-Prager Plasticity

Stresses in the medium are constrained by yield condition:

$$F(\sigma, \gamma^p) = \bar{\tau} - (\sigma_Y + h\gamma^p) \leq 0, \quad (2)$$

where γ^p is the hardening parameter (equivalent plastic strain) and h is the hardening modulus.

$$\bar{\tau} = \sqrt{s_{ij}s_{ij}/2} \quad (3)$$

is the second invariant of the deviatoric stress $s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$.

Drucker-Prager Plasticity

The yield stress

$$\sigma_Y = -(\sigma_{kk}/3)\sin(\phi) + c\cos(\phi) \tag{4}$$

where c is the cohesion and ϕ is the internal friction angle.

The flow rule given by $\dot{\epsilon}^p_{ij} = \lambda P_{ij}$ where $\lambda = \sqrt{2\dot{e}^p_{ij}\dot{e}^p_{ij}}$ is the deviatoric plastic strain rate, such that

$$\gamma^p(t) = \int_0^t \lambda(s)\, ds. \tag{5}$$

Drucker-Prager Plasticity

P_{ij} is the plastic strain rate tensor, given by

$$P_{ij} = s_{ij}/(2\bar{\tau}) + (\beta/3)\delta_{ij} \tag{6}$$

where β determines degree of plastic dilatancy.

Drucker-Prager Plasticity

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These details can be accounted for by expressing the stress-strain relationship in incremental form:

$$d\sigma = C^{ep}(\sigma) : d\epsilon \tag{7}$$

where the elastoplastic tangent stiffness tensor C^{ep} is a nonlinear function of stress.

Summary of Equilibrium Equation for Elastic vs. Plastic:

Elastic:

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)=0$$

For shear modulus μ.

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Elastic:

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For shear modulus μ.

Plastic:

$$\frac{\partial}{\partial x}\left(C_{11}^{ep}(\sigma)\frac{\partial du}{\partial x} + C_{12}^{ep}(\sigma)\frac{\partial du}{\partial z}\right) + \frac{\partial}{\partial z}\left(C_{21}^{ep}(\sigma)\frac{\partial du}{\partial x} + C_{22}^{ep}(\sigma)\frac{\partial du}{\partial z}\right) = 0$$

For elastoplastic moduli $C^{ep}(\sigma)$ and displacement increment du. We denote this

$$E\left[C^{ep}(\sigma) du\right] = 0 \qquad (8)$$

Elastoplastic moduli for Antiplane

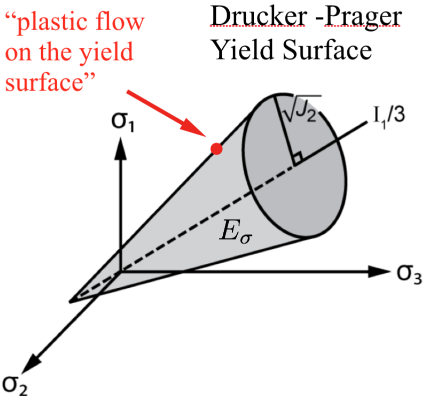
$$C_{11}^{ep} = \begin{cases} \mu & \text{if } \lambda = 0, \\ \mu - \frac{\mu\sigma_{xy}^2/\bar{\tau}^2}{1+h/\mu} & \text{if } \lambda > 0, \end{cases}$$

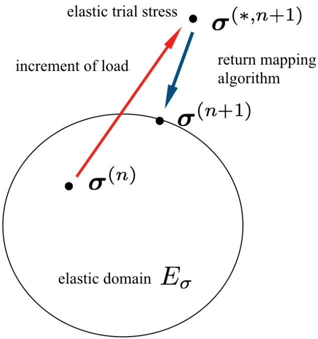
$$C_{22}^{ep} = \begin{cases} \mu & \text{if } \lambda = 0, \\ \mu - \frac{\mu\sigma_{yz}^2/\bar{\tau}^2}{1+h/\mu} & \text{if } \lambda > 0, \end{cases}$$

and

$$C_{12}^{ep} = C_{21}^{ep} = \begin{cases} 0 & \text{if } \lambda = 0, \\ -\frac{\mu\sigma_{xy}\sigma_{yz}/\bar{\tau}^2}{1+h/\mu} & \text{if } \lambda > 0. \end{cases}$$

The Return-Mapping Algorithm





Drucker-Prager reduces to von-Mises in antiplane strain. The return map is the closest point projection onto the yield surface.

Fully discretized problem

The fully discrete, equilibrium equation can be expressed

$$E\left[C^{ep}(\sigma^{n+1})du^{n+1}\right] = b^{n+1}, \tag{9}$$

and is a nonlinear function of du^{n+1}. Vector b^{n+1} stores all the information about the boundary conditions.

Time-Stepping Method

Assume the system is equilibrated at time t^n and that the stresses satisfy the yield condition.

1. Explicitly integrate to obtain slip δ^{n+1} and state θ^{n+1}. This yields the incremental boundary conditions:

$$
\begin{aligned}
du(0, z, t^{n+1}) &= (\delta(z, t^{n+1}) - \delta(z, t^n))/2 \\
du(L_x, z, t^{n+1}) &= V_p(t^{n+1} - t^n)/2 \\
d\sigma_{yz}(y, 0, t^{n+1}) &= 0 \\
d\sigma_{yz}(y, L_z, t^{n+1}) &= 0.
\end{aligned}
$$

The task is then to determine du^{n+1} that satisfies equilibrium, with consistent stresses.

Time-Stepping Method, cont'd

2. Set $k = 0$, let $du^{(k)}$ be the initial (elastic) guess for the displacement increment du^{n+1}. Correct this by the Newton procedure (detailed shortly).

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3. Compute shear stress on the fault $\tau_{qs}^{n+1} = \sigma_{xy}^{n+1}\Big|_{x=0}$.

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4. Equate shear stress with frictional strength $\tau_{qs}^{n+1} - \eta V^{n+1} = \sigma_n f(V^{n+1}, \theta^{n+1})$ and solve for slip velocity V^{n+1}. Return to step 1.

Newton method

Nonlinear equation:

$$E\left[C_{ep}(\sigma^{n+1})du^{n+1}\right] = 0. \qquad (10)$$

Recall that we have an initial, elastic guess $du^{(k)}$.

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Recall that we have an initial, elastic guess $du^{(k)}$.

2a. Compute the strain increments:

$$\begin{aligned} d\epsilon_{xy}^{(k)} &= \frac{1}{2}\frac{\partial du^{(k)}}{\partial x}, \\ d\epsilon_{yz}^{(k)} &= \frac{1}{2}\frac{\partial du^{(k)}}{\partial z} \end{aligned}$$

2b. Compute elastic trial state and use return mapping algorithm to compute consistent stresses $\sigma_{xy}^{n+1,(k)}, \sigma_{yz}^{n+1,(k)}$.

Newton procedure, cont'd

2c. Check if equilibrium equation

$$E\left[C^{ep}(\sigma^{n+1,(k)})du^{(k)}\right] = b^{n+1} \tag{11}$$

is met. If so, end.

Newton procedure, cont'd

2c. Check if equilibrium equation

$$E\left[C^{ep}(\sigma^{n+1,(k)})du^{(k)}\right] = b^{n+1} \tag{11}$$

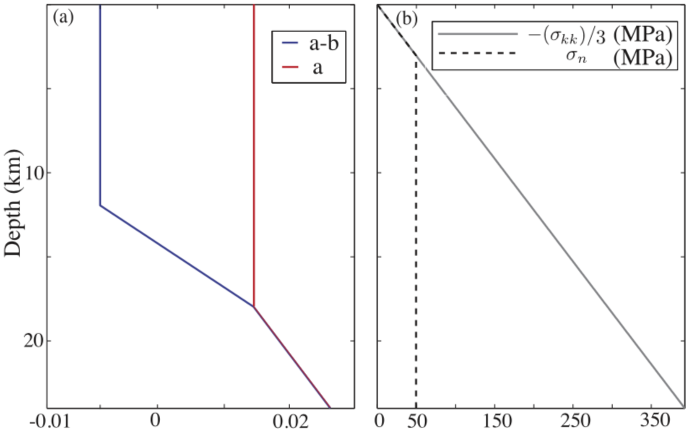
is met. If so, end.

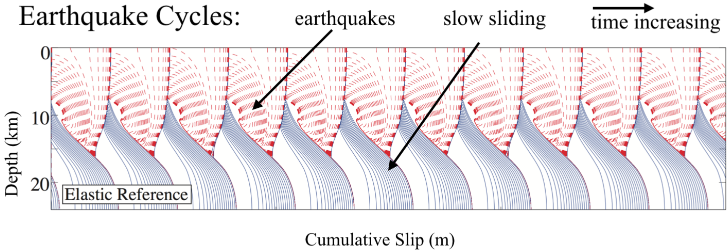
If not, set $k = k + 1$, and solve

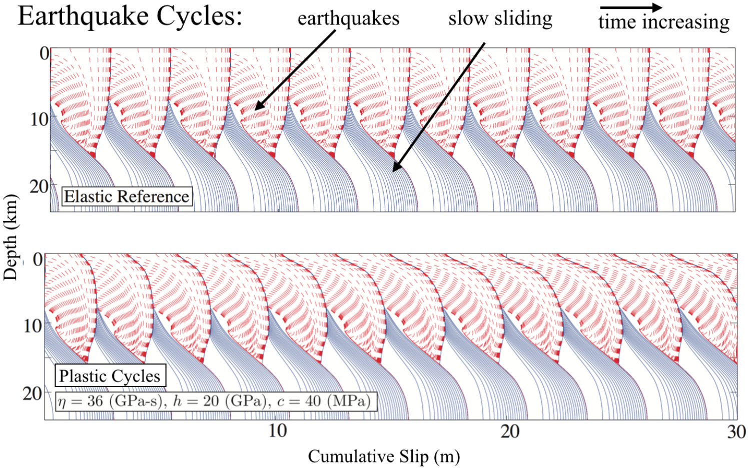
$$E\left[C^{ep}(\sigma^{n+1,(k)})du^{(k+1)}\right] = b^{n+1} \tag{12}$$

for the update $du^{(k+1)}$ and return to step 2a, iterating until equilibrium is met.

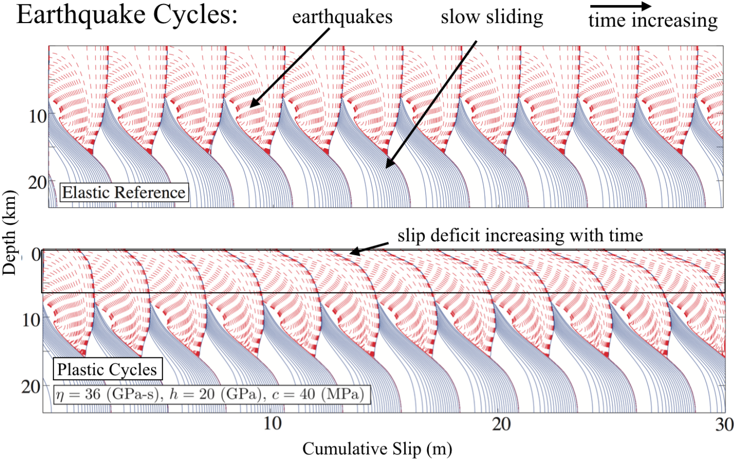
Rate-and-state parameters, normal stress



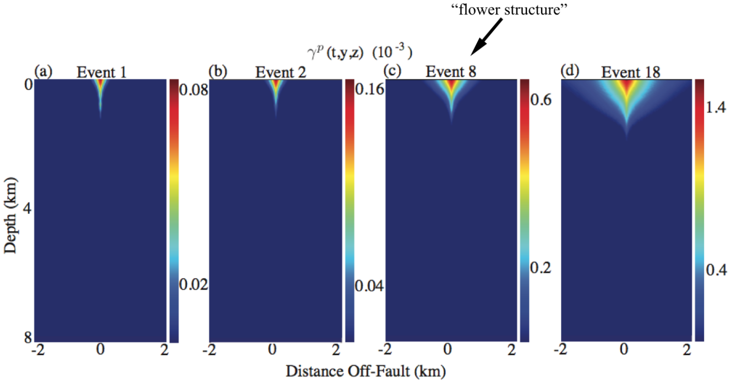

Earthquake Cycles:
earthquakes
slow sliding
time increasing
0
10
20
Depth (km)
Elastic Reference
Cumulative Slip (m)


Earthquake Cycles:
earthquakes
slow sliding
time increasing
0
10
20
Elastic Reference
Depth (km)
0
10
20
Plastic Cycles
$\eta = 36$ (GPa-s), $h = 20$ (GPa), $c = 40$ (MPa)
10
20
30
Cumulative Slip (m)

Earthquake Cycles:

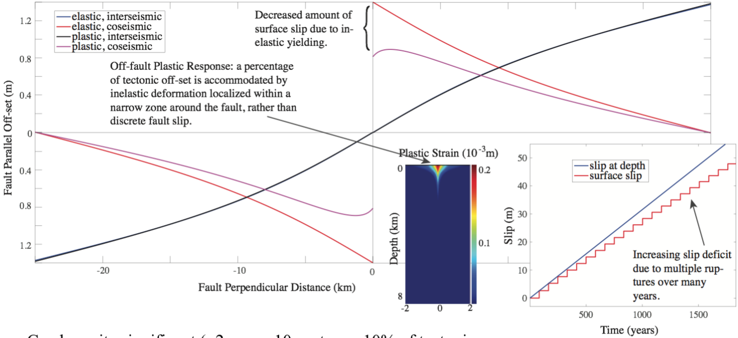


Evolution of damage zones:



A distribution of plastic strain corresponds to a region of permanently fractured rocks.

How much off-set is accommodated by plastic strain?



Can be quite significant (~2 m per 10 ruptures, 10% of tectonic deformation budget). If SSD deficit of 3-19% exists (Xu et al., 2016), then some of this can be attributed to plastic deformation.

Some benefits and drawbacks

- Methods capable of integrating through periods characterized by varying time scales, account for both rate-independent and visco-plasticity.
- With this plastic description, plastic response limited to coseismic phase.
- Computationally expensive. Currently implemented in serial code, with direct solver. Need to re-compute matrix factorization at every time step and every Newton iteration.

Thank you.

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